Homework Problem #14

(Exercise 3.3) Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and $y \in P$. Prove that $d \in \mathbb{R}^n$ is a feasible direction at $y$ if and only if $Ad = 0$ and $d_i \geq 0$ whenever $x_i = 0$.

Claim: Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and $y \in P$. Vector $d \in \mathbb{R}^n$ is a feasible direction at $y$ if and only if $Ad = 0$ and $d_i \geq 0$ whenever $x_i = 0$.

Definition 1. Let $x$ be an element of polyhedron $P$. A vector $d \in \mathbb{R}^n$ is said to be a feasible direction at $x$ if there exists a positive scalar $\theta$ for which $x + \theta d \in P$.

Proof. Suppose $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$.

Suppose $d \in \mathbb{R}^n$ is a feasible direction at $y \in P$. Then there exists $\theta > 0$ so that $y + \theta d \in P$. We show that $Ad = 0$ and that $d_i \geq 0$ whenever $y_i = 0$. $b = A(y + \theta d) = Ay + \theta Ad = b + \theta Ad$, so $Ad = 0$. Also, $y + \theta d \geq 0$, so if $y_i = 0$ then $d_i \geq 0$, but if $y_i > 0$ then there is no sign restriction on $d_i$.

Next, suppose $y \in P$, $d \in \mathbb{R}^n$, $Ad = 0$ and $d_i \geq 0$ whenever $y_i = 0$. We show that there exists $\theta > 0$ so that $y + \theta d \in P$. First, notice that $y + \theta d$ satisfies the equality constraints for any $\theta > 0$. $A(y + \theta d) = Ay + \theta Ad = Ay = b$. Next, we show that for sufficiently small $\theta$, $y + \theta d \geq 0$. If $y_i = 0$ then we know that $d_i \geq 0$ and so $y_i + \theta d_i \geq 0$ for any $\theta > 0$. If $y_i > 0$ the $y_i + \theta d_i \geq 0$ whenever $\theta \leq y_i/|d_i|$. So, we take $\theta = \min_{i,y_i>0} \{y_i/|d_i|\} > 0$. □