Homework Problem #12

Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0\}$, where the rows of $A$ are linearly independent.

1. Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.

Suppose $A$ is $m \times n$. We have $\text{rank}(A) = m$. Each basis consists of $m$ basic variables and $n - m$ zero-valued nonbasic variables. If $n = m$ then there is only one possible basis. So, we consider the case $n > m$. Suppose, without loss of generality, that the nonbasic variables of the first basis are $x_1, x_2, \ldots, x_{n-m}$ and the nonbasic variables of the second basis are any $n - m$ variables, not all the same as the first. If both bases lead to the same basic solution, then $x_1 = x_2 = \cdots = x_{n-m} = 0$ for both basic solutions. So, the basic solution has more than $n - m$ zero-valued variables. Thus, the basic solution is degenerate.

2. Consider a degenerate basic solution. Is it true that it corresponds to two distinct bases? Prove or give a counterexample.

False. Not all degenerate basic solutions correspond to two distinct bases. This can happen when the choice of potential basic variables leads to a system of equations with no solution, or when $n = m$ and the single feasible point is degenerate. As an example of the latter, consider $P = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1, \ x_1 - x_2 = 1\}$. Because $n = m = 2$, there is only one basis $B = \{1, 2\}$, with corresponding basic (feasible) solution $x = (1, 0)$. This point is degenerate because it has more than $n - m = 0$ zeros. We can also see that this point is degenerate because there are three active constraints in $n = 2$.

A more rich example is the standard form polyhedron $P = \{s \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1, \ x_1 = 0, \ x \geq 0\}$ which has two basic solutions, both of which are degenerate and each of which corresponds to a single basis.

3. Suppose that a basic solution is degenerate. Is it true that there exists an adjacent basic solution which is degenerate? Prove or give a counterexample.

False. The example in part 2 has a single degenerate basic solution, so there are no adjacent basic solutions. A more rich example is the standard form polyhedron $P = \{x \in \mathbb{R}^3 \mid x_1 - x_2 = 0, \ x_1 + x_2 + x_3 = 1, \ x \geq 0\}$, which has two basic solutions which are adjacent but only one of which is degenerate.