Homework Problem #11

Consider the standard form polyhedron \( P = \{ x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0 \} \), where \( A \) is \( m \times n \) with linearly independent rows. For each of the following statements, state whether it is true or false. If true, provide a formal proof. If false, provide a clear counterexample.

1. If \( n = m + 1 \), then \( P \) has at most two basic feasible solutions.
2. The set of all optimal solutions is bounded.
3. At every optimal solution, no more than \( m \) variables can be positive.
4. If there is more than one optimal solution, then there are infinitely many optimal solutions.
5. If there are several optimal solutions, then there exist at least two basic feasible solutions which are optimal.
6. Consider the problem of minimizing \( f(x) = \max \{ c^T x, d^T x \} \) over the set \( P \). If this problem has an optimal solution, then it must have an optimal solution which is an extreme point of \( P \).

Solution.

1. True.

Claim: The standard form polyhedron \( P = \{ x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0 \} \), where \( A \) is \( m \times n \) with linearly independent rows and \( n = m + 1 \), has at most two basic feasible solutions.

Proof. Suppose \( A \) is an \( m \times n \) matrix with linearly independent rows and \( n = m + 1 \). Let \( P \) be the polyhedron \( P = \{ x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0 \} \). We show that \( P \) has at most two extreme points, so that by Theorem 2.3, \( P \) has at most two basic feasible solutions. Because \( \text{rank}(A) = m \), the set \( \{ x \in \mathbb{R}^n \mid Ax = b \} \) is a hyperplane of dimension \( n - m = (m+1) - m = 1 \), that is, a line. The feasible set \( P \) is then the intersection of this line with the positive orthant \( \{ x \in \mathbb{R}^n \mid x \geq 0 \} \). This is either a line segment with two vertices or a ray with one vertex. Thus, we have at most two vertices and at most two basic feasible solutions. \( \square \)

2. False. An unbounded ray or face of the feasible region can consist of optimal solutions. For example,

\[
\begin{align*}
\min_x \quad & z = x_1 - x_2 \\
\text{s.t} \quad & x_1 - x_2 = 1 \\
& x \geq 0 \\
& x \in \mathbb{R}^2
\end{align*}
\]

is a standard form linear program for which every feasible point is optimal.

3. False. Consider the example linear program in part 2. The vector \((5, 4)\) is optimal with \( m = 1 \), but two variables are positive.
4. True.

Claim: If the standard form linear program $\min z = c^T x \text{ s.t. } x \in P$ (where $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, $A$ is an $m \times n$ matrix with linearly independent rows) has more than one optimal solution, then it has infinitely many optimal solutions.

Proof. Suppose $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, $A$ is an $m \times n$ matrix with linearly independent rows. Furthermore, suppose the linear program $\min z = c^T x \text{ s.t. } x \in P$ has distinct optimal solutions $y$ and $z$. Then the infinite set of points $W = \{w \in \mathbb{R}^n \mid w = ay + (1 - a)z, a \in [0, 1]\}$ are also optimal. To show this, note that $W$ is convex, so $W \subset P$ and $c^T w = c^T y$ for all $w \in W$. (I leave it to the reader to show this.) \hfill \Box

5. False. Consider the example linear program in part 2. The set of optimal solutions is a ray extending from $(1,0)$ infinitely in the direction $d = (1,1)$. There is only one basic feasible solution.

6. False. Consider the standard form polyhedron $P = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 2, x \geq 0\}$ and the objective $f(x) = \max \{x_1, x_2\}$. The optimal solution is $x^* = (1,1)$, which is not a vertex of $P$, and $z^* = 1$. 