Homework Problem #4

(Exercise 1.8) Consider a road divided into \( n \) segments which is illuminated by \( m \) lamps. Let \( p_j \) be the power of the \( j^{th} \) lamp. Suppose also that the illumination of the \( i^{th} \) road segment is \( I_i = \sum_{j=1}^{n} a_{ij} p_j \), where the \( a_{ij} \) are known constants (determined by the relative positions of the road segments and lamps). The road department has specified a desired illumination \( J_i \) for each road segment. How can we choose lamp powers so that the road is well-illuminated? Provide a reasonable linear programming formulation.

(Solution #1) Consider the problem of minimizing the power delivered to the lamps subject to achieving the road department specified illumination. Let \( p \in \mathbb{R}^m \) be the decision variables. Then the objective is to minimize the total power: \( Z = \sum_{j=1}^{m} p_j \).

Achieving proper illumination is \( I_i \geq J_i \) for all \( i \), where \( I_i = \sum_{j=1}^{n} a_{ij} p_j \). The powers \( p_j \) must be non-negative as well. We have the linear program:

\[
\begin{align*}
\min \quad & Z = \sum_{j=1}^{m} p_j \\
\text{s.t.} \quad & I_i \geq J_i \quad i = 1, 2, \ldots, n \\
& I_i = \sum_{j=1}^{n} a_{ij} p_j \quad i = 1, 2, \ldots, n \\
& p \geq 0 \\
& p \in \mathbb{R}^m, \quad I \in \mathbb{R}^n
\end{align*}
\]

or equivalently,

\[
\begin{align*}
\min \quad & Z = \sum_{j=1}^{m} p_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} p_j \geq J_i \quad i = 1, 2, \ldots, n \\
& p \geq 0 \\
& p \in \mathbb{R}^m
\end{align*}
\]
Notice that if we define $A$ as the matrix with entries $a_{ij}$, then we have the equivalent matrix form

\[
\begin{array}{c}
\min_{P} z = c^T p \\
\text{s.t. } Ap \leq b \\
\quad p \geq 0 \\
\quad p \in \mathbb{R}^m
\end{array}
\]

\[
c = [1 \ 1 \ \ldots \ 1]^T \\
A = [a_{ij}] \\
b = [J_1 \ J_2 \ \ldots \ J_n]^T
\]

(Solution #2) Consider the problem of creating illumination that most closely matches the road department specified illumination. That is, choose power $p$ so that $I_i \approx J_i$ for each road segment $i$. In particular, we might minimize the "worst" difference:

\[
\min_{P} z = \max_i \left\{ |I_i - J_i| \right\}
\]

\[
\text{s.t. } I_i = \sum_{j=1}^{n} a_{ij} p_j \quad i = 1, 2, \ldots, n
\]

This objective is not a linear function. We can reformulate the problem using a new variable, say $s$, that measures the worst difference:

\[
\begin{array}{c}
\min_{P, S} z = s \\
\text{s.t. } s \geq \sum_{j=1}^{n} a_{ij} p_j - J_i \quad i = 1, 2, \ldots, n \\
\quad s \geq -\sum_{j=1}^{n} a_{ij} p_j + J_i \quad i = 1, 2, \ldots, n \\
\quad p \in \mathbb{R}^m, \quad p \geq 0 \\
\quad s \in \mathbb{R}
\end{array}
\]
This LP has \( m+1 \) decision variables (\( \mathbf{p} \in \mathbb{R}^m \), \( \mathbf{s} \in \mathbb{R} \)) and \( 2n \) inequality constraints and \( m \) sign constraints. This formulation works because \( \mathbf{s} \) is greater than (or equal to) all of the absolute differences and we seek to minimize that value by choosing \( \mathbf{p} \). A slight rearrangement is:

\[
\begin{align*}
\min_{\mathbf{s}, \mathbf{p}} \quad & z = \mathbf{s} \\
\text{s.t.} \quad & \mathbf{s} - \sum_{j=1}^{n} a_{ij} \mathbf{p}_j \geq -J_i \quad \forall i = 1, 2, \ldots, n \\
& \mathbf{s} + \sum_{j=1}^{n} a_{ij} \mathbf{p}_j \geq J_i \\
& \mathbf{s}, \mathbf{p} \geq 0 \\
& \mathbf{s} \in \mathbb{R} \\
& \mathbf{p} \in \mathbb{R}^m
\end{align*}
\]

which we can also write in matrix form:

\[
\begin{align*}
\min_{\mathbf{x}} \quad & z = \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & \mathbf{B} \mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq 0 \\
& \mathbf{x} \in \mathbb{R}^{m+1} \\
\end{align*}
\]

\[
\begin{bmatrix}
1_n & -\mathbf{A} \\
1_n & \mathbf{A}
\end{bmatrix}, 
\begin{bmatrix}
-\mathbf{J} \\
\mathbf{J}
\end{bmatrix}
\]

\[
\mathbf{x} = \begin{bmatrix} \mathbf{s} & \mathbf{p}_1 & \mathbf{p}_2 & \ldots & \mathbf{p}_m \end{bmatrix}^T \\
\mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}^T
\]

\( 1_n \) is the column vector of \( n \) ones. 
\( \mathbf{J} \) is the column vector of \( J_i \) values. 
\( \mathbf{A} \) is the \( nxm \) matrix of \( a_{ij} \) values.
(Solution #3) The previous solution was "soft" in that we only requires $I_i \approx J_i$, but the optimal solution could have $I_i < J_i$ for some $i$. So, we could instead try:

$$\min_{p} z = \max_{i} \sum_{i} I_i - J_i \mid$$

s.t. $I_i \geq J_i \quad i=1,2,\ldots,3$

$p \geq 0$

$p \in \mathbb{R}^m$

This solution can be developed similarly to the previous solution.