Tuesday March 10

1. Schedule this week In Vancouver Wed & Thr

2. The Two-Phase Simplex Method and Homework #19, #20.

3. Project: Define your optimization problem!
   → I am here to help!

4. Today's topic: Duality Theory

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* More on the two-phase Simplex Method (when and how to implement an auxiliary problem)
  * Use auxiliary LP whenever an immediate initial basis is not evident.
  * Also known as the Two-Phase Method.
  * Problem #19
    - Inequality form, add 3 new slack variables
      - If the slack variables indicate the basis then the basis is infeasible.
      - The basic variables are \( x_5 = 2 \), \( x_7 = 2 \), \( x_9 = 0 \).

- Degenerate point & infeasible
  - When the slack variables were included \( x_7 \) was added
    \( x_5 \) & \( x_9 \) were subtracted
  - If we follow the recipe above we end up adding one new slack variable, say \( x_7 \)
    and end up with an improper tableau (does not have 3 basic columns).

- (a) Should read “Add a new non-basic variable is not positive.”

For example, a non-basic column

\[
\begin{array}{c|c|c|c|c}
 & x_1 & x_2 & x_3 & \text{RHS} \\
\hline
x_1 & 0 & 1 & -1 & 0 \\
\hline
x_2 & 1 & -1 & 0 & 0 \\
\hline
x_3 & -1 & 0 & 1 & 0 \\
\end{array}
\]
When using an auxiliary problem (Two-Phase Method), the initial goal is to see if (if when) the auxiliary slack variables can be non-basic.

\[
\begin{align*}
\text{max } w &= -x_6 - x_7 - x_8 \\
\text{s.t. } &
\end{align*}
\]

where \( W = x_6 + x_7 + x_8 \) non basic

Initial basis for problem 20

\[
\begin{align*}
X_6 = X_7 = X_8 &= 0 \\
\Rightarrow &
\end{align*}
\]

Decision variables

Even if \( x_6 = x_7 = x_8 = 0 \) they may not be non-basic variables

What do you do if \( x_6 = x_7 = x_8 = 0 \) but the current basis includes one or more of \( x_0, x_1, x_2 \)? (because we are at a degenerate point)

- You are at a degenerate point (optimal point)
- You can pivot to change the basis
- You can always pivot so that \( x_6, x_7, x_8 \) are non-basic

Sometimes you may need to pivot using a non-basic variable whose reduced cost is positive (i.e., as long as the step size is zero).

**Duality Theory**

Consider the standard form LP

\[
\begin{align*}
\text{min } z &= c^T x \\
\text{s.t. } & A x = b \\
& x \geq 0
\end{align*}
\]

Question: Is it easy to find a good lower bound on \( z^* \)?

Consider the related problem

\[
\begin{align*}
\text{min } c^T x + P^T (b - Ax) \\
\text{s.t. } & x \geq 0
\end{align*}
\]

let \( g(p) \) be the optimal cost of this relaxed problem for any choice of vector \( p \).

\[
\begin{align*}
g(p) &= \min_{x \geq 0} c^T x + P^T (b - Ax) \\
&= c^T x^* + P^T (b - Ax^*) \\
&= c^T x^*
\end{align*}
\]

(\( g(p) \leq c^T x^* \geq z^* \)) \( g(p) \) is the lower bound on \( z^* \).

What is the best (or tightest) lower bound?

\[
\begin{align*}
\text{max } w &= g(p) \\
\text{s.t. } & p \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{But } g(p) &= \min_{x \geq 0} c^T x + P^T (b - Ax) \\
&= P^T b + \min_{x \geq 0} (c^T x - P A x) \\
&= P^T b + \min_{x \geq 0} (c^T - P A) x
\end{align*}
\]

15 (b) is positive then \( x = 0 \)

16 (c) not positive then
Consider an inequality form primal problem
\[ \min z = c^T x \]
\[ \text{s.t.} \quad A x \leq b \]
\[ x \geq 0 \]

\text{(PRIMAL)}

\text{(DUAL)}
\[ \max w = b^T p \]
\[ \text{s.t.} \quad A^T p \leq c \]
\[ p \geq 0 \]

\[ \min \frac{1}{2} p^T A p + c^T p \]
\[ \text{s.t.} \quad p \leq 0 \]
\[ p \geq 0 \]

\text{Notice: } A, b, c \text{ are the same for both primal & dual.}

Does every LP have a dual? Yes.

For each primal-dual pair \( z^* = w^* \) (if they exist)

"The dual of the dual is the primal"