• **Homework Status**
  • HW 12, 13 now due Saturday PM
  • Still doing well
  • Read posted solutions

• **Project Status**
  • Email me the status
  • Don't promise results!
  • You have freedom to modify "their problem".

• **Participation**
  • I like to see you in class
  • and prepared.
Simplex Method Idea

- Given a feasible basis (basic feasible solution)
  \[ X_N = 0 \quad X_B = B^{-1}b \]

- If all reduced costs are non-negative then the current bfs is optimal.
  \[ \bar{c}_B = 0 \quad \bar{c}_N = c_N^T - c_B B^{-1} A_N \]

- If not optimal, choose \( j \) where \( \bar{c}_j < 0 \) (selecting a non-basic variable and basic direction)
  \[ d_j = 1 \quad d_k = 0 \quad d_B = -B^{-1} A_j \]

- Find the step length \( \theta^* \) in direction \( d \)
  \[ \theta^* = \max_{\theta \geq 0} \{ x + \theta d \in P \} \]
  \[ = \min_{i=1,2,...,m} \left\{ \frac{-X_B(i)}{d_B(i)} \right\} \] (ratio test)

- Find the new non-basic index \( B(\ell) \) and new basis
  \[ d_B(\ell) < 0 \quad \Rightarrow \quad X_B(\ell) + \theta^* d_B(\ell) = 0 \]
  Swap (pivot) \( j \leftrightarrow B(\ell) \)
  New basis \( B(1), B(2), \ldots B(\ell-1), j, B(\ell+1), \ldots B(m) \)

- Go to step 1.
We really only need to keep track of $\vec{c}$, $x_B$ and $d_B$. (and $z$).

Create a Simplex Tableau:

<table>
<thead>
<tr>
<th></th>
<th>$-C_B^T B^{-1} b$</th>
<th>$C^T - C_B^T B^{-1} A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-z$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B^{-1} b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B^{-1} A$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Reduced costs array that contains $d_B$

Negative of objective

Basic variable values

$$-C_B^T x_B$$

$$C_1, C_2, \ldots, C_n$$

Reduced costs

![Matrix](matrix.png)

If $j \in N$ then $-B^{-1} A_j = d_B$.

The $j$th basic direction.
Consider the example problem with basis \( \{2, 3\} \)

\[
\begin{align*}
\min \ z &= C^T x \\
\text{s.t.} \quad A x &= b \\
 x &\geq 0 \\
 x &\in \mathbb{R}^4
\end{align*}
\]

\[
A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}
\]

\[
C_B^T = \begin{bmatrix} 2 & 1 \\ 1 & -\frac{1}{3} \end{bmatrix}, \quad x_B^T = [x_2 \ x_3]
\]

\[
x_B = B^{-1} b = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}
\]

\[
-Z = -C_B^T x_B = -\begin{bmatrix} 2 & 1 \\ 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = -\frac{10}{3}
\]

\[
B^{-1} A = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ \frac{2}{3} & 0 & 1 & \frac{4}{3} \end{bmatrix}
\]

\[
\tilde{C} = C^T - C_B B^{-1} A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 \frac{1}{3} \ 0 \ 0 \ 4 \frac{1}{3} \end{bmatrix}
\]

\[
\begin{array}{rrrrr|c}
\hline
& x_1 & x_2 & x_3 & x_4 & \text{red costs} \\
\hline
-\frac{10}{3} & -\frac{1}{3} & 0 & 0 & 4 \frac{1}{3} & \frac{4}{3} = \frac{1}{3} x_1 + x_2 + 3 x_3 - \frac{1}{3} x_4 \\
\hline
4 \frac{1}{3} & \frac{1}{3} & 1 & 0 & -\frac{1}{3} & \\
2 \frac{1}{3} & 0 & 1 & 4 \frac{1}{3} & \\
\hline
\end{array}
\]

Basic feasible solution \( x = (0, \frac{4}{3}, \frac{3}{3}, 0) \)

See next page for explanation.
The initial Tableau:

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>$-\frac{10}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

- **This Tableau is not optimal:** At least one reduced cost is negative.
- $X_N = 0$ (nonbasic variables) \( \Rightarrow X_1 = X_4 = 0 
- $X_B = (\frac{4}{3}, \frac{2}{3})$
- The current bfs is $X = (0, \frac{4}{3}, \frac{2}{3}, 0)$, $Z = \frac{10}{3}$
- Each row (except top row) is simply a restatement of an equality constraint: EX: $\frac{4}{3} = \frac{1}{3}X_1 + 1X_2 + 0X_3 + \frac{1}{3}X_4$.
- Basic columns are marked as $\star$. Notice $\bar{c}_B = 0$.
- Nonbasic columns are marked as $\star$. In general, $\bar{c}_N \neq 0$.
- Because $\bar{c}_i < 0$ (i.e., there is a neighbor vertex with an improved objective. We have the direction $d_1 = 1$, $d_4 = 0$, $d_2 = -\frac{1}{3}$, $d_3 = -\frac{2}{3}$.
- The ratio test shows that as $x_i$ increases from a value of zero, $x_3$ decreases (first) to a value of zero. This happens at step size $\theta^* = \min \left\{ \frac{4}{3}, \frac{2}{3} \right\} = 1$.
- The pivot is then to swap $X_1$ into the basis and $X_3$ out of the basis.
- The pivot location is circled as $\circ$.
- To perform the pivot (swap), perform row operations to make the $j=1$ column a basic column:
  - The pivot position has value 1.
  - The remaining entries in the column have value 0.

- The new tableau is

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-Z =$</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$x_2 =$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$x_1 =$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
</tr>
</tbody>
</table>

This tableau is optimal ($C \geq 0$) with basic feasible solution

$$x = (1, 1, 0, 0) \quad z = 3$$