Things coming due:

today: HW problem #6
Thursday: HW problem #7
Tuesday (11th): HW problems #8, #9

Other Reminders

make a solid contact plan and get it approved by me.
Definition 3. A set $S \subseteq \mathbb{R}^n$ is bounded if there exists $K \in \mathbb{R}$ such that $\|x\| < K$ for all $x \in S$.

Definition 4. A set $S \subseteq \mathbb{R}^n$ is convex if for any $x, y \in S$ and any $a \in [0, 1]$, $ax + (1-a)y \in S$.

Example #1: Prove that every hyperplane in $\mathbb{R}^n$ is convex.

Example #2: Prove that every halfspace in $\mathbb{R}^n$ is not bounded.

Group Tasks:

4. Prove that the feasible region of any linear program is convex.

Reword Definition 3:
(a) "A set $S$, a subset of vectors with $n$ real entries, is bounded if there exists a real number $K$ such that the norm of a vector $x$ is less than $K$ for every vector $x$ in the set $S$.
(b) "A set $S$ is bounded if some ball centered at the origin contains $S$.
(c) "A set $S$ is bounded if a ball contains it."

Reword Definition 4:
(a) "A set $S$ is convex if every line segment with endpoints in $S$ is also completely in $S"
DEF: (hyperplane)

DEF: (convex set)

Every hyperplane in $\mathbb{R}^n$ is convex.

Proof:

Suppose $H = \{ x \in \mathbb{R}^n | a^T x = b \}$, $z, y \in H$, $\alpha \in [0, 1]$.

Define $\omega = \alpha z + (1 - \alpha) y$.

We demonstrate that $H$ is convex by showing that $\omega \in H$, that is $a^T \omega = b$.

Notice $a^T \omega = a^T[\alpha z + (1 - \alpha)y]$

$= \alpha a^T z + (1 - \alpha) a^T y$

$= \alpha b + (1 - \alpha) b$

$= b$

So $\omega \in H$ and thus, $H$ is convex.

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Every halfspace in \( \mathbb{R}^n \) is not bounded.

Proof: (by contradiction) Suppose, by way of contradiction, that halfspace \( H = \{ x \in \mathbb{R}^n \mid a^T x \geq b \} \) is bounded.

We will demonstrate that for any \( K > 0 \) there exists \( x \in H \) such that \( \|x\| > K \), a contradiction, so that \( H \) is not bounded.

By assumption, \( H \) is bounded so there exist \( K > 0 \) so that \( \|x\| < K \) for all \( x \in H \). By definition there exists at least one non-zero element of \( a \), say \( a_k \).

If \( a_k > 0 \) then consider \( x \) such that \( x_k = M, M > K, M > \frac{1}{a_k} \) and \( x_{i \neq k} = 0 \). we have

\[
    a^T x = a_k x_k > \frac{1}{a_k} \cdot M \geq b, \quad \text{so } x \in H.
\]

But \( \|x\| = M > K \), a contradiction.

A similar argument holds for \( a_k < 0 \).

Thus, every halfspace in \( \mathbb{R}^n \) is not bounded. \( \square \)
The feasible region of any linear program is convex.

Proof: Consider the feasible region \( P = \{x \in \mathbb{R}^n \mid Ax \leq b \} \) of a general linear program. We demonstrate that \( P \) is convex by showing that for arbitrary \( y, z \in P \) and \( a \in [0,1] \), \( w = ay + (1-a)z \in P \). For \( w \in P \) we must show \( Aw \leq b \). Observe:

\[
Aw = A \left[ ay + (1-a)z \right] \\
= aAy + (1-a)Az \\
\leq ab + (1-a)b \\
= b.
\]

So \( w \in P \). Thus \( P \) is convex. \( \square \)