Announcements / Reminders

- Office Hours MTW 11-12:30
  Neill 228
  Zoom meeting # 952 228 1142

- No exams = make the HW excellent

- Read 1.1-1.4 over the next 7-10 days

- Overleaf Links
  - A few did not include (speed.png)
  - Send me the link only once

- Feel free to email me
Some optimization problems are simpler:

Objective function:
\[
\min_x Z = C^T x
\]

Linear program (LP):
\[
\begin{align*}
\text{s.t.} & \quad a_i^T x \geq b_i \quad i = 1, 2, \ldots, m, \\
& \quad a_j^T x = b_j \quad j = 1, 2, \ldots, m, \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]

Linear optimization problem:
\[
\Omega = \{ x \in \mathbb{R}^n \mid a_i^T x \geq b_i \quad i = 1, 2, \ldots, m, \\
& \quad a_j^T x = b_j \quad j = 1, 2, \ldots, m \}
\]

What we often want is a Standard Form LP:

\[
\begin{align*}
\min_x Z &= c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]

\( A \) is \( m \times n \) matrix
\( m \leq n \)
\( \text{rank}(A) = m \)

Note: Every LP can be reformulated to be in Standard Form.
EXAMPLE:

\[
\begin{align*}
\min_x & \quad z = 3x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \geq 4 \\
& \quad x_1 - x_3 \leq 6 \\
& \quad 2x_2 + x_3 = 2 \\
& \quad x_1 \geq 0 \\
& \quad x_3 \geq 0 \\
& \quad x \in \mathbb{R}^3
\end{align*}
\]

This is a LP.
- objective is linear
- ineq. const. linear
- eq. const. linear
- decision variables are real valued

\[
\begin{align*}
\min_x & \quad z = 3x_1 + x_2 + 0x_3 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \geq 4 \\
& \quad -x_1 + x_3 \leq -6 \\
& \quad x_1 \geq 0 \\
& \quad x_3 \geq 0 \\
& \quad 2x_2 + x_3 = 2 \\
& \quad x \in \mathbb{R}^3
\end{align*}
\]

rewrite

\[
\begin{align*}
\min_x & \quad z = c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad A_e x = b_e \\
& \quad x \in \mathbb{R}^3
\end{align*}
\]

\[
\begin{align*}
c &= [3, 1, 0]^T \\
A &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \\
b &= \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix} \\
A_e &= \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}, \\
b_e &= \begin{bmatrix} 2 \end{bmatrix}
\end{align*}
\]

linear program LP

rewrite

equivalent matrix form LP

\[
\begin{align*}
\min_x & \quad z = c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad A_e x = b_e \\
& \quad x \in \mathbb{R}^3
\end{align*}
\]

\[
\begin{align*}
c &= [3, 1, 0]^T \\
A &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \\
b &= \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix} \\
A_e &= \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}, \\
b_e &= \begin{bmatrix} 2 \end{bmatrix}
\end{align*}
\]
Next, reformulate the example LP into standard form

- Inequality constraints
  
  \[(x_1 + x_2 + x_3 \geq 4)\]

  is equivalent to:
  
  \[
  \begin{cases}
  x_1 + x_2 + x_3 - x_4 = 4 \\
  x_4 \geq 0 \\
  \end{cases}
  \]

  \[x_4\] is called a slack variable

- Similarly
  
  \[(x_1 - x_3 \leq 6)\]

  is equivalent to
  
  \[
  \begin{cases}
  x_1 - x_3 + x_5 = 6 \\
  x_5 \geq 0 \\
  \end{cases}
  \]

- \[x_2\] is unrestricted (unrestricted in sign)

  replace any urs variable with the difference of 2 non-negative variables.

  \[
  \begin{cases}
  x_2 = x_6 - x_7 \\
  x_6 \geq 0 \\
  x_7 \geq 0 \\
  \end{cases}
  \]

  Then, substitute \(x_6 - x_7\) for \(x_2\) everywhere
We have the equivalent SFLP

\[
\begin{align*}
\min \quad & z = 3x_1 + x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 - x_4 = 4 \\
& x_1 - x_3 + x_5 = 6 \\
& 2x_2 + x_3 = 2 \\
& x_1, x_3, x_5, x_4 \geq 0 \\
& x \in \mathbb{R}^5
\end{align*}
\]

\[
\begin{align*}
\min \quad & z = 3x_1 + x_6 - x_7 \\
\text{s.t.} \quad & x_1 + x_3 - x_9 + x_6 - x_7 = 4 \\
& x_1 - x_3 + x_5 = 6 \\
& x_3 + 2x_6 - 2x_7 = 2 \\
& x \in \mathbb{R}^6 \\
& x \geq 0
\end{align*}
\]

\[
\begin{align*}
\min \quad & z = c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0 \\
& x \in \mathbb{R}^6
\end{align*}
\]

\[
A = \begin{bmatrix}
1 & 1 & -1 & 0 & 1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & -2
\end{bmatrix}
\]

\[
b = \begin{bmatrix}
4 \\
6 \\
2
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
x_1 \\
x_3 \\
\vdots \\
x_7
\end{bmatrix}
\]
Bananas sell for $6 per pound and rice $1 per pound. Bananas contain 5 units of vitamin A and 1 unit of vitamin C per pound.

Rice contains 1 unit of each vitamin per pound.

What is the least expensive scheme that provides a person with 3 units vitamin A and \( \frac{3}{2} \) units vitamin C?

**Decision Variables:**

- Let \( x_1 \) = (number of pounds of bananas to buy)
- \( x_2 \) = (number of pounds of rice)

(This works, but there may be other reasonable choices)

\( x \in \mathbb{R}^2 \)

**Objective Function:**

Let \( z \) be the cost in dollars

\[ z = 6x_1 + x_2 \]

**Constraints:**

- Vitamin A: \( 5x_1 + x_2 \geq 3 \)
- Vitamin C: \( x_1 + x_2 \geq \frac{3}{2} \)

**Sign Constraints:**

\( x \geq 0 \)
solution: Our decision variables are the quantities of each type of food to purchase (and eat). Let

\[ x_1 = \text{ (the pounds of bananas to purchase)} \]
\[ x_2 = \text{ (the pounds of rice to purchase)} \]

We will consider possible non-integer quantities, so \( x \in \mathbb{R}^2 \).

The objective is to minimize the cost of items purchased. Let \( z \) be the cost in dollars. The per-pound cost of each item is provided. We have then

\[ z = 6x_1 + x_2 \]

The vitamin A and vitamin C content of each item (per pound) is also given. We must ensure eating enough of each vitamin, and we will assume that extra vitamin content is acceptable. We have

\[ 5x_1 + x_2 \geq 3 \quad \text{(Vitamin A)} \]
\[ x_1 + x_2 \geq \tfrac{3}{2} \quad \text{(Vitamin C)} \]

Finally, we must purchase non-negative quantities of each item: \( x \geq 0 \). Collecting these results into a LP:

\[
\begin{align*}
\text{min } z &= 6x_1 + x_2 \\
\text{s.t. } 5x_1 + x_2 &\geq 3 \\
&\quad x_1 + x_2 \geq \tfrac{3}{2} \\
&\quad x \geq 0 \\
&\quad x \in \mathbb{R}^2
\end{align*}
\]