MATH 420 – Example Proofs

Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$ and let X and Y be subsets of V. Prove that if $X \subseteq Y$ then $\textbf{\textit{span}}\ X \subseteq \textbf{\textit{span}}\ Y$.

Proof. Suppose $X \subseteq Y$. We will show that an arbitrary vector in $\textbf{\textit{span}}\ X$ is also in $\textbf{\textit{span}}\ Y$. First, note that $\textbf{\textit{span}}\ X \neq \emptyset$ even if $X = \emptyset$.
Let $v = a_1x_1 + a_2x_2 + \cdots + a_kx_k \in \textbf{\textit{span}}\ X$ where $x_1, x_2, \cdots, x_k \in X$ and scalars $a_1, a_2, \cdots, a_k \in \mathbb{F}$. Since $X \subseteq Y$, $x_1, x_2, \cdots, x_k \in Y$ so $v \in \textbf{\textit{span}}\ Y$. Thus, $\textbf{\textit{span}}\ X \subseteq \textbf{\textit{span}}\ Y$. □

Let $(\mathcal{F}, +, \cdot)$ be the vector space over $\mathbb{F}$, where $\mathcal{F}$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ and with addition and scalar multiplication defined by $(f + g)(x) = f(x) + g(x)$ and $(a \cdot f)(x) = af(x)$, respectively. Prove that the subset of even functions in $\mathcal{F}$, denoted $\mathcal{E}$, is a subspace of $\mathcal{F}$.

Proof. We show that subset $\mathcal{E}$ (a) is closed under addition, (b) is closed under scalar multiplication and (c) contains the zero vector of $\mathcal{F}$. Then by Theorem 5.1.1, $\mathcal{E}$ is a subspace of $\mathcal{F}$. Note that $\mathcal{E} = \{f \in \mathcal{F} \mid f(x) = f(-x)\}$. Let $f, g \in \mathcal{E}$ and $a \in \mathbb{F}$.

(a) $(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x)$.

(b) $(a \cdot f)(x) = af(x) = af(-x) = (a \cdot f)(-x)$.

(c) Let $0(x)$ be the zero function in $\mathcal{F}$. $0(x) = 0 = 0(-x)$.

□