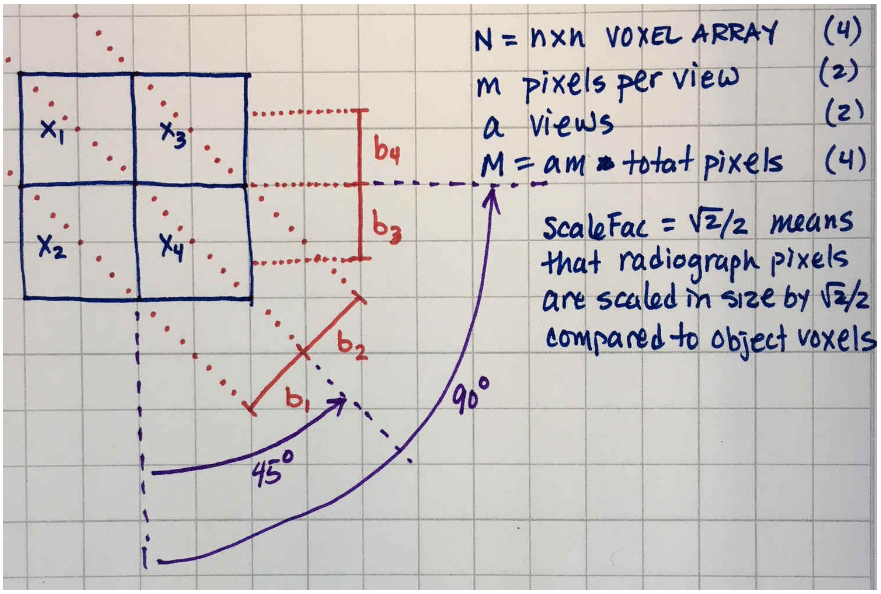
1. Sketch the geometric represenation of the radiographic scenario represented by the parameters: $n = 2$, $m = 2$, $a = 2$, $\theta_1 = 45°$, $\theta_2 = 90°$, $ScaleFac = \sqrt{2}/2$.



2. Compute the radiographic transformation T for this same scenario.

There are $N = 4$ object voxels and $M = 4$ radiograph pixels. So, T can be expressed as a 4×4 matrix, where T_{kj} is the fraction of voxel j intersecting with beam path k. The geometry show us that

$$T = \begin{bmatrix} 1/2 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \end{bmatrix}$$

Definition 1. *Let* $(V, +, \cdot)$ *and* $(W, \oplus, \odot)$ *be vector spaces over* $\mathbb{F}$. *We say that* $T : V \to W$ *is a* <u>*linear transformation*</u> *if for all* $v_1, v_2 \in V$ *and every* $\alpha \in \mathbb{F}$, $T(\alpha \cdot v_1 + v_2) = \alpha \odot T(v_1) \oplus T(v_2)$.

Definition 2. *Let* $(V, +, \cdot)$ *and* $(W, \oplus, \odot)$ *be vector spaces over* $\mathbb{F}$. *We say that* $T : V \to W$ *is the* <u>*zero transformation*</u> *if* $T(v) = 0$ *for all* $v \in V$.

Definition 3. *Let* $(V, +, \cdot)$ *be a vector space over* $\mathbb{F}$. *We say that* $T : V \to V$ *is the* <u>*identity transformation*</u> *if* $T(v) = v$ *for all* $v \in V$.

Let $V = \left\{ \begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} \subseteq M_{2\times 3}(\mathbb{R})$.

Define $h : V \to P_1(\mathbb{R})$ by $h \begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix} = ax + c$.

1. What is the domain of h?

   The domain of h is V. This is the vector space of all defined inputs to h.

2. What is the codomain of h?

   The codomain of h is $P_1(\mathbb{R})$. This is the vector space containing all possible output of h.

3. Is V a subspace of $M_{2\times 3}(\mathbb{R})$?

   V is a subspace of $M_{2\times 3}(\mathbb{R})$ if (a) it is closed under vector addition, (b) it is closed under scalar multiplication and (c) the zero vector of $M_{2\times 3}(\mathbb{R})$ is in V. Observe:

(a) Let $v = \begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix} \in V$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned} \alpha v &= \alpha \begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix} \\ &= \begin{pmatrix} \alpha a & \alpha b & \alpha c \\ 0 & \alpha(b-c) & 2\alpha a \end{pmatrix} \\ &= \begin{pmatrix} a' & b' & c' \\ 0 & b'-c' & 2a' \end{pmatrix} \in V \end{aligned}$$

where $a' = \alpha a$, $b' = \alpha b$ and $c' = \alpha c$. So, V is closed under scalar multiplication.

(b) (Use a similar argument to show closure under vector addition.)

(c) Let $v = \begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix} \in V$. If $a = b = c = 0$, then $v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is the zero vector in $M_{2\times 3}(\mathbb{R})$.

4. Find a basis for V.

Notice that an arbitrary vector in V can be written

$$v = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

So,

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

is a spanning set for V. B is also a basis for V because it is linearly independent (The reader should show this.)

5. What is the dimension of V?

dim $V = |B| = 3$.

6. What is the dimension of $P_1(\mathbb{R})$?

The standard basis for $P_1(\mathbb{R})$ is $S = \{1, x\}$, so dim $P_1(\mathbb{R}) = |S| = 2$.

7. What is the range of h?

   The range of the function h is the set of all possible outputs, a subset of the codomain $P_2(\mathbb{R})$. We can show that for an arbitrary vector $w \in P_1(\mathbb{R})$, there exists a vector $v \in V$ such that $h(v) = w$, so that the range of h is $P_1(\mathbb{R})$. Let $w = \alpha x + \beta$. Then $v = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & -\beta & 2\alpha \end{pmatrix} \in V$ satisfies $h(v) = w$.

8. Is h a linear transformation?

   For h to be a linear transformation, it must satisfy the definition. In particular, for $v, w \in V$ and $\alpha \in \mathbb{R}$, $T(\alpha v + w) = \alpha T(v) + T(w)$. Let $v = \begin{pmatrix} a & b & c \\ 0 & b-c & 2a \end{pmatrix}$ and $w = \begin{pmatrix} d & e & f \\ 0 & e-f & 2d \end{pmatrix}$.

$$\begin{aligned} T(\alpha v + w) &= T\begin{pmatrix} \alpha a + d & \alpha b + e & \alpha c + f \\ 0 & \alpha(b-c) + (e-f) & 2\alpha a + 2d \end{pmatrix} \\ &= (\alpha a + d)x + (\alpha c + f) \\ &= \alpha(ax + c) + (dx + f) \\ &= \alpha T(v) + T(w). \end{aligned}$$

9. Give a zero transformation $z : V \to P_1(\mathbb{R})$.

   The zero transformation $z : V \to P_1(\mathbb{R})$ is the transformation defined by $z(v) = 0x + 0 = 0$. It is the transformation that maps any matrix in V to the zero polynomial in W.

10. Give $T_1 : V \to P_1(\mathbb{R})$ and $T_2 : P_1(\mathbb{R}) \to V$ such that $T_1 \circ T_2$ is the identity transformation.

11. Give $T_1 : V \to P_1(\mathbb{R})$ and $T_2 : P_1(\mathbb{R}) \to V$ such that $T_2 \circ T_1$ is the identity transformation.