Chapter 6 Exercise 16

A system of equations is called homogeneous if the right-hand-side values of each equation are zero. Show that every homogeneous system of equations has at least one solution.

In matrix form, a homogeneous system is written $Ax=0$ for some $m \times n$ coefficient matrix A and variable vector $x \in \mathbb{R}^n$. $x=0$ is a solution to any homogeneous system.

Exercise 39

Is it possible to write a linear combination (with nonzero coefficients) of u, v and w as a linear combination of just two of these vectors? Justify your response.

Yes. Consider the general linear combination $au+bv+cw$ and suppose $w = du+ev$ (a linear combination of u and v). Then $au+bv+cw = au+bv+c(du+ev) = (a+cd)u+(b+ce)v$.

Exercise 43

Can w be written as a linear combination of u and v where $u = 3x^2+x+2$, $v = x^2-2x+3$, $w = -x^2-1$. $u, v, w \in P_2(\mathbb{R})$.

Are there scalars $a, b \in \mathbb{R}$ so that $au+bv=w$?

$$a(3x^2+x+2)+b(x^2-2x+3) = (-x^2-1).$$

Equating like polynomial terms yields the system of equations

$$\begin{aligned} 3a+b &= -1 \\ a-2b &= 0 \\ 2a+3b &= -1 \end{aligned}$$

with solution $a = -2/7$, $b = -1/7$. Ans: Yes! with

$$w = (-2/7)u + (-1/7)v.$$

Chapter 7 Exercise 13

Show, with justification, that $\operatorname{span}(X \cup Y) = (\operatorname{span} X) \cup (\operatorname{span} Y)$ is, in general, false.

Consider an example in $\mathbb{R}^2$. Let $X = \{(1,0)\}$, $Y = \{(0,1)\}$.
Then, $X \cup Y = \{(1,0), (0,1)\}$ and $\operatorname{span}(X \cup Y) = \mathbb{R}^2$.
However, $\operatorname{span} X = \{(a,0) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$ (the x-axis)
and $\operatorname{span} Y$ is the y-axis. So, $(\operatorname{span} X) \cup (\operatorname{span} Y)$ is the set of all points either on the x-axis or on the y-axis, or in particular, not all of $\mathbb{R}^2$.

Exercise 14 Show that $(\operatorname{span} X) \cup (\operatorname{span} Y)$ is not necessarily a subspace.

Consider the same example from Exercise 13.
$(\operatorname{span} X) \cup (\operatorname{span} Y)$ is not a subspace because it is not closed under vector addition. For example, $(a,0)$ and $(0,b)$ are both in $(\operatorname{span} X) \cup (\operatorname{span} Y)$ but $(a,0)+(0,b) = (a,b)$ is not in $(\operatorname{span} X) \cup (\operatorname{span} Y)$ unless $a=b=0$.

Exercise 15

Show, with justification, that $\operatorname{span}(X \cap Y) = (\operatorname{span} X) \cap (\operatorname{span} Y)$ is, in general, false.
Let $X = \{(1,0), (0,1)\}$ so $\operatorname{span} X = \mathbb{R}^2$ and let $Y = \{(1,1), (-1,1)\}$
so $\operatorname{span} Y = \mathbb{R}^2$ as well. So $(\operatorname{span} X) \cap (\operatorname{span} Y) = \mathbb{R}^2$.
However, $X \cap Y = \phi$ and $\operatorname{span}(X \cap Y) = \{0\}$.