Appendix E

Proof Techniques

In this Appendix, we will cover techniques and etiquette commonly seen in proofs. Before beginning, we will give some rules of logic to help understand what follows.

E.1 Logic

In this section, we layout standard rules of mathematical logic and the interpretation of words such as ‘or’ and ‘and.’ Before getting too deep, we need to define what we mean when we say ‘statement.’

Definition E.1.1. *A statement is a sentence with a truth value.*

Before giving examples, we need understand ‘truth value.’

Note A statement has a truth value if it is either true or it is false. There are many statements that we do not have proof for their truth values, but it is clear that there is a truth value. For example,

> “There is life in another galaxy.”

is a statement. We do not have the technology to check all galaxies for life and we haven’t found life yet. But, we do know that it must be either true or false.

Example E.1.1. *Here are a few more statements.*

1. *If* $x = 0$ *then* $3x + 1 = 1$.
2. *Whenever* $3x^2 + 2 = 5$ *is true,* $x = 6$ *is also true.*
3. *Everyone who reads this book is taking a Linear Algebra course.*
4. *Some cats are pets.*

As you can see, not all of the statements above are true. Indeed, statements 1 and 4 are true, but statements 2 and 3 are false.

Example E.1.2. *A statement cannot be a phrase, exclamation, an incomplete idea, nor an opinion. For example*

A. *Blue*
B. *Cats are the best pets.*
C. *Wow!*

The sentence E.1.2(B) is not considered a statement. You may want to argue that it is true, but it is only an opinion.

We can combine statements to make new statements using conjunctions (such as *and,*, *or*, or *if... then*) This is where the rules of logic become necessary.

When 'and' is used to connect two statements, the new statement is true only when both original statements are true. When 'or' is used to connect two statements, the new statement is false only when both are original statements are false. In the truth table below, we have indicated the truth for the various cases that occur when connecting two statements using 'If...then...' Given two statements, P and Q, we use the notation given in the table below.

Notation	Meaning
T	true
F	false
$P \wedge Q$	P and Q,
$P \vee Q$	P or Q, and
$P \Rightarrow Q$	P implies Q or If P then Q.
$\sim P$	not P

The truth values (T or F) for various logic cases are shown in Table E.1. Each row of the table shows the logical truth value for conjunctions and implications formed from P and Q based on the given truth values in the

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \wedge \sim Q$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T	T	T	F	T
T	F	F	T	F	T	F	T	F
F	T	T	F	F	T	T	F	T
F	F	T	T	F	F	T	F	T

Table E.1: A truth table for conjunctions and implications formed using statements P and Q.

first two columns. Notice that the last column of the table above shows that the truth values of $P\wedge \sim Q$ are just the opposite of $P \Rightarrow Q$. This will be very useful later when we discuss proof by contradiction. Notice also that $P \Rightarrow Q$ and $\sim Q \Rightarrow\sim P$ have the same truth value. This will be very important when discussing the contrapositive.

E.2 Proof structure

In this section, we layout what we consider a good structure for a proof. This should only serve as a starting point for your proof writing. Later, as you become more comfortable writing proofs, it is expected that you will add your own voice to your proofs without losing the necessary components for a good proof.

In order to not belabor this, we draw out the necessary components here.

Statement This is the statement that you will be proving.

Proof. ⟵Every proof should begin with this. In cases of particular types of proofs, you may indicate the technique you will be using. More on that later.
Hypotheses: Here, you list all the relevant assumptions. You use words like *Assume that...* or *Suppose...* or *Let...*
Plan: Here, you give the reader an idea about how the proof will work. *We will show that...*
Proof Body: Here is where you begin listing statements that link to definitions, theorems, algebra, arithmetic,... these will all lead to the result being proved
Conclusion: Here you state what you just proved.
Every proof should end with some sort of symbol indicating your proof is done. One very common example is the symbol here.⟶ □

Now, we move to various proof techniques.

E.3 Direct Proof

In this section, we will show some examples of writing direct proofs. Many times, it is more proper to give a direct proof of a statement than any other type of proof. This is because direct proofs tend to be more straightforward and easier to follow. Of course, this is not always the case and we discuss other proof techniques in upcoming sections.

In the following examples, we will add comments in blue to emphasize important features.

Proofs use theorems and definitions to make a clear argument about the truth of another statement. To illustrate this, we will use the following definition.

Definition E.3.1. *A number $n \in \mathbb{Z}$ is* <u>*even*</u> *if there exists $k \in \mathbb{Z}$ so that $n = 2k$.*

Example E.3.1. *If $n \in \mathbb{Z}$ is even, then n^2 is even.*

Proof. Suppose $n \in \mathbb{Z}$ is even.	*We always start by telling the reader what we are assuming.*
We will find $k \in \mathbb{Z}$ so that $n^2 = 2k$.	*We follow up with our goal or our plan.*
We know that $n = 2m$ for some $m \in \mathbb{Z}$.	*Write what our assumptions mean. This is what we have to work with.*
Notice $n^2 = 4m^2 = 2(2m^2)$.	*We use algebra to find $k = 2m^2$.*
Since $2m^2 \in \mathbb{Z}$, we can use the definition of even to say that n^2 is even. □	*We always end a proof stating the result.*

Notice above, that all our assumptions came from the hypothesis of the statement. Notice also that the proof ended with the conclusion of the statement. In the next example, we will refrain from adding the blue comments. Try to find the same elements in this next proof.

Example E.3.2. *If n and m are both even integers, then $n + m$ is an even integer.*

Proof. Suppose n and m are even integers. We will show that there is a $k \in \mathbb{Z}$ so that $n + m = 2k$. We know that there are integers x and y so that $n = 2x$ and $m = 2y$. Then $n + m = 2x + 2y = 2(x + y)$. Let $k = x + y \in \mathbb{Z}$, then $n + m = 2k$. Thus, $n + m$ is an even integer. □

E.4 Contrapositive

In this section, we discuss the contrapositive of a statement. We will show how and when to use the contrapositive to prove a statement. First, we define the contrapositive.