OCTOBER 10 2018 MATH 420
(TOPICS REVIEW DAY)

① Let $V = \{ax^2+2bx+3c \in P_2(\mathbb{R}) \mid 2a-b=0, a,b,c \in \mathbb{R}\}$

(Be able to show that V is a subspace of $P_2(\mathbb{R})$. We did not do this in class.)

Find a basis for V. Notice that

$$V = \{ax^2+4ax+3c \in P_2(\mathbb{R}) \mid a,b,c \in \mathbb{R}\}$$
$$= \{a(x^2+4x)+c(3) \mid a,c \in \mathbb{R}\}$$

Since $\{x^2+4x, 3\}$ is a L.I. spanning set for V, it is a basis for V.

② Let $X = \{(a,b,a+b) \in \mathbb{R}^3 \mid a,b \in \mathbb{R}\}$

Find a basis for X. Notice that

$$X = \{a(1,0,1)+b(0,1,1) \mid a,b \in \mathbb{R}\}$$

Since $\{(1,0,1),(0,1,1)\}$ is a L.I. spanning set for X, it is a basis for X.

③ ⓐ Is $x^2+x+1 \in V$?

No. All vectors in V must have the second coefficient 4 times the value of the first coefficient.

ⓑ Is $-x^2-4x+7 \in V$?

Yes, for the same reasoning.

③ ⓒ Is $(1,2,4) \in X$?

No. All vectors in X have third entry equal to the sum of the first two.

ⓓ Is $(0,1,1) \in X$?

Yes, for the same reason.

④ Suppose $B = \{v_1, v_2, v_3\}$ is L.I. Show that $C = \{v_1-v_2, v_1+v_2, v_3\}$ is L.I.

Most people answered this question "incorrectly" using an argument like the following:

Consider the linear dependence relation for C:

$$a(v_1-v_2)+b(v_1+v_2)+c(v_3)=0.$$

If $a=b=c=0$ is the only solution, then C is linearly independent. We have

$$(a+b)v_1+(b-a)v_2+(c)v_3=0$$
$$\Rightarrow \left\{\begin{array}{l} a+b=0 \\ b-a=0 \\ c=0 \end{array}\right\} \Rightarrow a=b=c=0$$

The "problem" with the above argument is that "$\Rightarrow$" is not explained. If "$\Rightarrow$" means "comparing like terms" then the argument is incorrect! "$\Rightarrow$" actually means "because B is L.I., this linear dependence relation has only the trivial solution:"

⑤ Determine if the following transformations are linear.

ⓐ $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)x^2+(2c-d)x+2$

No. If a transformation is linear, it must be true that $T(0_V)=0_W$ (where $T: V \to W$). In this case:

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0x^2+0x+2 = 2 \neq 0.$$

ⓑ $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (abcd)x^2$

No. We show by counterexample that $T(\alpha v) \neq \alpha T(v)$. Let $\alpha = 3$ and $v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$T(\alpha v) = T\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right) = 81x^2$$

$$\alpha T(v) = 3T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 3x^2.$$

⑤ ⓒ $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$

$T[f(x)] = f'(x)$.

Yes. T is linear. We show that $T[(af+g)(x)] = aT[f(x)]+T[g(x)]$ for all $f,g \in P_n(\mathbb{R})$ and $a \in \mathbb{R}$.

$$T[(af+g)(x)] = T[af(x)+g(x)]$$
$$= af'(x)+g'(x)$$
$$= aT[f(x)]+T[g(x)]$$

Be able to prove that for a linear transformation $T: V \to W$, $T(0_V)=0_W$.