Name: **SOLUTIONS**

WSU ID#: ______________________________

MATH 420 Fall 2018
Midterm Exam #1

Instructions:

1. Write your name and WSU ID# at the top of this page.
2. Your work is to be your own work, unassisted by any other person or their work.
3. Do not use any materials other than those provided with this exam.
4. Do not include scratchwork. Please use the supplied blank paper to formulate your thoughts.
5. Always justify your work. Always explain linear algebra concepts central to the question.
6. This exam includes all definitions and theorems which you may require.
7. There are six questions.
8. The exam is worth 70 points. The first five questions are worth 10 points each. The last question has multiple parts and is worth 20 points.
9. You must use appropriate mathematical notation and language to receive full credit.

1	2	3	4	5	6	

Definition 1. *Consider a set of objects* X *with scalars taken from* $\mathbb{R}$*, and operations of addition* $(+)$ *and scalar multiplication* $(\cdot)$ *defined on* X*. We say that* X *is* *closed under addition* *if* $x + y \in X$ *for all* $x, y \in X$*. We say that* X *is* *closed under scalar multiplication* *if* $\alpha \cdot x \in X$ *for each* $x \in X$ *and each* $\alpha \in \mathbb{R}$*.*

Definition 2. *Consider a set* V *over a field* $\mathbb{F}$ *(either* $\mathbb{R}$ *or* Z_2*) with given definitions for addition* $(+)$ *and scalar multiplication* $(\cdot)$*.* V *with* $+$ *and* $\cdot$ *is called a* *vector space over* $\mathbb{F}$ *if for all* $u, v, w \in V$ *and for all* $\alpha, \beta \in \mathbb{F}$*, the following ten properties hold.*

(P1) $u + v \in V$.

(P2) $\alpha \cdot v \in V$.

(P3) $u + v = v + u$.

(P4) $(u + v) + w = u + (v + w)$.

(P5) $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$.

(P6) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.

(P7) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.

(P8) There exists $0 \in V$*, called the additive identity, such that* $0 + v = v + 0 = v$.

(P9) There exists $z \in V$*, such that* $v + z = 0$*, called the additive inverse of* v.

(P10) The scalar set $\mathbb{F}$ *has an identity element, denoted* 1*, for scalar multiplication:* $1 \cdot v = v$.

Theorem 1. *If* x, y*, and* z *are vectors in a vector space* $(V, +, \cdot)$ *and* $x + z = y + z$*, then* $x = y$*.*

Theorem 2. *Let* $(V, +, \cdot)$ *be a vector space over* $\mathbb{F}$*.* $0 \cdot x = 0$ *for each* $x \in V$*.*

Theorem 3. *Let* $(V, +, \cdot)$ *be a vector space over* $\mathbb{R}$ *with additive identity element* 0*.* $\alpha \cdot 0 = 0$ *for each* $\alpha \in \mathbb{R}$*.*

Theorem 4. *Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. $(-\alpha) \cdot x = -(\alpha \cdot x) = \alpha \cdot (-x)$ for each $\alpha \in \mathbb{F}$ and each $x \in V$.*

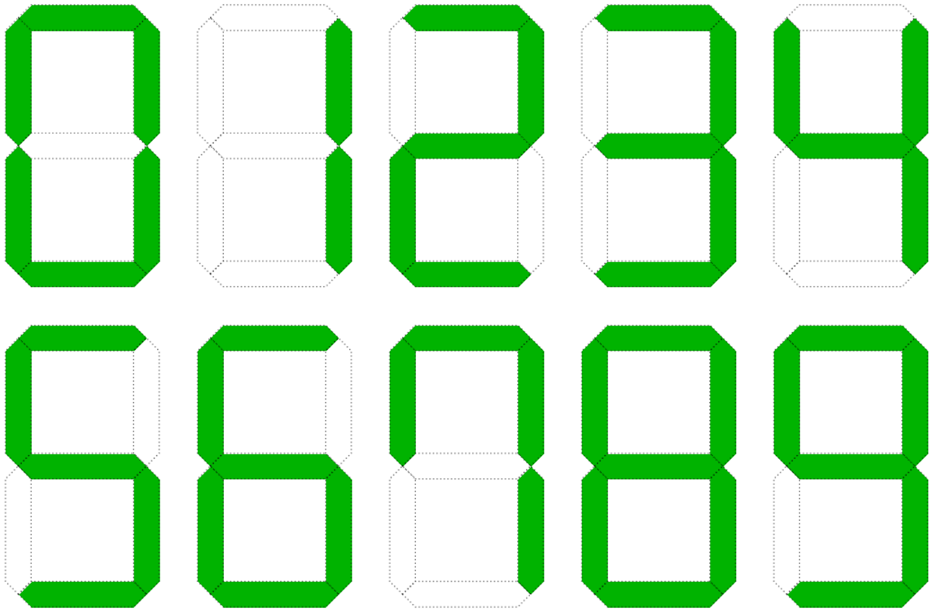


Figure 1: The ten digits of a standard 7-bar LCD display, as vectors in $\mathcal{D}(Z_2)$, where Z_2 is the field that includes only the scalar values 0 and 1.

Definition 3. *Let $(V, +, \cdot)$ be a vector space over a field $\mathbb{F}$. If $W \subseteq V$ (W is a subset of V), then we say that W is a subspace of $(V, +, \cdot)$ whenever $(W, +, \cdot)$ is also a vector space.*

Theorem 5. *Let $(V, +, \cdot)$ be a vector space over a field $\mathbb{F}$ and $W \subseteq V$. $(W, +, \cdot)$ is a subspace of V if and only if $0 \in W$ and W is closed under (the inherited operations of) vector addition and scalar multiplication.*

Theorem 6. *Let W_1 and W_2 be any two subspaces of a vector space $(V, +, \cdot)$. Then $W_1 \cap W_2$ is a subspace of $(V, +, \cdot)$.*

Theorem 7. *Let W_1 and W_2 be any two subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

Theorem 8. *Let W_1 and W_2 be subspaces of a vector space $(V, +, \cdot)$. Then $W_1 + W_2$ is a subspace of $(V, +, \cdot)$.*

Definition 4. *Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. Given a finite set of vectors $v_1, v_2, \cdots, v_k \in V$, we say that the vector $w \in V$ is a linear combination of $v_1, v_2, \cdots, v_k$ if $w = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$ for some scalar coefficients $a_1, a_2, \cdots, a_k$.*

Definition 5. *Let $(V, +, \cdot)$ be a vector space and let $X \subseteq V$. The span of the set X, denoted $\textbf{span}\, X$, is the set of all linear combinations of the elements of X. In addition, $\textbf{span}\, \emptyset \equiv \{0\}$.*

Definition 6. *We say that the (possibly infinite) set S spans (or generates) a set W if $W = \textbf{span}\, S$. In this case, we also say that the vectors in S span the set W.*

Definition 7. *We say that a (possibly infinite) set S is a spanning set (or generating set) for W if $W = \textbf{span}\, S$.*

Theorem 9. *Let X be a subset of vector space $(V, +, \cdot)$. Then $\textbf{span}\, X$ is a subspace of V.*

Theorem 10. *Let X and Y be subsets of a vector space $(V, +, \cdot)$. Then the following statements hold.*

(a) $\textbf{span}\,(X \cap Y) \subseteq (\textbf{span}\, X) \cap (\textbf{span}\, Y)$.

(b) $\textbf{span}\,(X \cup Y) \supseteq (\textbf{span}\, X) \cup (\textbf{span}\, Y)$.

(c) *If $X \subseteq Y$, then $\textbf{span}\, X \subseteq \textbf{span}\, Y$.*

(d) $\textbf{span}\,(X \cup Y) = (\textbf{span}\, X) + (\textbf{span}\, Y)$.

1. Consider f, g, h, vectors in a vector space $(V, +, \cdot)$ over $\mathbb{F}$.

   (a) Explain how would determine if h can be written as a linear combination of f and g.

   h can be written as a linear combination of f and g if there exist scalars $a, b \in \mathbb{F}$ such that $h = a \cdot f + b \cdot g$.

   (b) Explain how you would know if $h \in \textbf{span}\ \{f, g\}$.

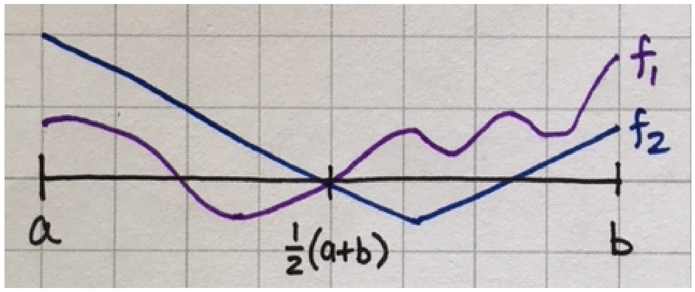
   h is in the span of $\{f, g\}$ if h can be written as a linear combination of f and g.

2. Thought question: Why might it be useful to have a spanning set for a subspace of a vector space? You may think in terms of a specific application or in more abstract terms.

   There are many possible good responses. Answers should touch on the key idea that a spanning set forms a sort of catalog from which any vector in the subspace can be generated. The vectors in a subspace may be of importance in an application (e.g. potential solutions to a problem) but there may be infinitely many. A spanning set can be a few representative vectors which can generate *any* subspace vector.

3. Consider the vector space $(\mathcal{F}, +, \cdot)$ over $\mathbb{R}$ consisting of functions $f : [a, b] \to \mathbb{R}$ with definitions $(f + g)(x) = f(x) + g(x)$ and $(a \cdot f)(x) = af(x)$. Let $\mathcal{G}$ be the subset of $\mathcal{F}$ defined by $\mathcal{G} = \left\{ f \in \mathcal{F} \mid f\left(\frac{1}{2}(a + b)\right) = 0 \right\}$.

   (a) Provide a clearly labeled sketch of two representative elements of $\mathcal{G}$.



   (b) Show that $(\mathcal{G}, +, \cdot)$ is a subspace of $(\mathcal{F}, +, \cdot)$. You ***do not*** need to write a formal proof; however, you ***do*** need to explain your reasoning.

   Because $\mathcal{G}$ is a subset of vector space $\mathcal{F}$, we need only show (a) $\mathcal{G}$ is closed under addition, (b) $\mathcal{G}$ is closed under scalar multiplication and (c) the zero vector of $\mathcal{F}$ is in $\mathcal{G}$. Let $f_1, f_2 \in \mathcal{G}$, $a \in \mathbb{R}$.

   (a) We show that $(f_1 + f_2) \in \mathcal{G}$ by showing $(f_1 + f_2)(\frac{1}{2}(a + b)) = 0$.

$$(f_1 + f_2)(\tfrac{1}{2}(a + b)) = f_1(\tfrac{1}{2}(a + b)) + f_2(\tfrac{1}{2}(a + b)) = 0 + 0 = 0.$$

   (b) We show that $(a \cdot f_1) \in \mathcal{G}$ by showing $(a \cdot f_1)(\frac{1}{2}(a + b)) = 0$.

$$(a \cdot f_1)(\tfrac{1}{2}(a + b)) = af_1(\tfrac{1}{2}(a + b)) = a(0) = 0.$$

   (c) The zero vector in $\mathcal{F}$ is the zero function $0(x) = 0$. Notice, $0(\frac{1}{2}(a + b)) = 0$. Thus, $0(x) \in \mathcal{G}$.

4. Show that $\mathbb{R}^2$ is not a vector space over $\mathbb{R}$ if vector addition, $\oplus$, is defined by $(a, b) \oplus (c, d) = (ac, b + d)$. Show this without knowing a specific definition of scalar multiplication. Be sure and clearly state your argument.

We show that under the given definition of addition, the property of Theorem 1 does not hold, so $\mathbb{R}^2$ cannot be a vector space over $\mathbb{R}$ in this situation. Let $u = (0, 2)$, $v = (1, 2)$ and $w = (2, 2)$. Notice:

$$u \oplus v = (0, 2) \oplus (1, 2) = ((0)(1), 2 + 2) = (0, 4),$$

$$u \oplus w = (0, 2) \oplus (2, 2) = ((0)(2), 2 + 2) = (0, 4).$$

We see that $u \oplus v = u \oplus w$ but $v \neq w$. So, by Theorem 1, $\mathbb{R}^2$ is not a vector space over $\mathbb{R}$ with $\oplus$ defined as above.

5. Provide a formal proof of Theorem 3 using only the provided definitions and theorems *listed before Theorem 3.*

Proof. Consider any $x \in V$.

$$\begin{aligned} \alpha \cdot 0 + \alpha \cdot x &= \alpha \cdot (0 + x) && (P6) \\ &= \alpha \cdot x && (P8) \\ &= 0 + \alpha \cdot x && (P8) \end{aligned}$$

Thus, by Theorem 1, $\alpha \cdot 0 = 0$. □

6. Consider the vector space of 7-bar LCD images, $\mathcal{D}(Z_2)$. Let d_2 and d_3 be the images of the digits 2 and 3, respectively.

   (a) Explicitly write out the following sets, giving *each* image in the sets:

      i. $D_2 = \textbf{span}\ \{d_2\}$ and $D_3 = \textbf{span}\ \{d_3\}$

      Let z be the image with no lit bars (the additive identity vector in the vector space).

$$D_2 = \{0 \cdot d_2, 1 \cdot d_2\} = \{z, d_2\}$$

$$D_3 = \{0 \cdot d_3, 1 \cdot d_3\} = \{z, d_3\}$$

      .

      ii. $U = \textbf{span}\ \{d_2, d_3\}$

      Let $v = d_2 + d_3$ be the image with only the two lower vertical bars lit.

$$U = \{0 \cdot d_2 + 0 \cdot d_3, 1 \cdot d_2 + 0 \cdot d_3, 0 \cdot d_2 + 1 \cdot d_3, 1 \cdot d_2 + 1 \cdot d_3\}$$
$$= \{z, d_2, d_3, v\}.$$

      iii. $W = D_2 + D_3$.

$$\begin{aligned} W &= \{z + z, d_2 + z, z + d_3, d_2 + d_3\} \\ &= \{z, d_2, d_3, v\} \\ &= U. \end{aligned}$$

      iv. $Q = D_2 \cap D_3$.

$$Q = \{z, d_2\} \cap \{z, d_3\} = \{z\}$$

      .

   (b) Is $U = D_2 \oplus D_3$? Justify.

   $U = W = D_2 + D_3$ and $D_2 \cap D_3 = Q = \{z\}$, so $U = D_2 \oplus D_3$.