(Wed Sept 12)

Here are the big ideas from class today.

The following are equivalent questions:

(a) Is vector v in span $\{x, y, z\}$?

(b) Can vector v be written as a linear combination of x, y and z?

(c) Are there constants $a, b, c \in \mathbb{F}$ such that $v = ax + by + cz$?

"Equivalent" means that either the answer to <u>all</u> questions is "yes" or the answer to <u>all</u> questions is "no".

The following are equivalent questions:

(a) Does $\{x, y, z\}$ span V?

(b) Are $x, y, z \in V$ <u>and</u> can any arbitrary $v \in V$ be written $v = ax + by + cz$ for some $a, b, c \in \mathbb{F}$?

(c) Are $x, y, z \in V$ <u>and</u> can every $v \in V$ be written as a linear combination of x, y and z?

We also noted that a spanning set for a vector space can have few or many vectors – as long as it spans V.

Here is one of our examples:

(1) Let $V = P_3(\mathbb{R})$. Is $v = x^2 - x^3$ in span $\{x^2, 2x - x^2, x + x^3\}$?

We want to know if $a, b, c \in \mathbb{R}$ exist so that

$$(x^2 - x^3) = a(x^2) + b(2x - x^2) + c(x + x^3).$$

It might be "easy" to notice that c must equal -1 and this leads to $a = 3/2$, $b = 1/2$. Since there is a solution, v is in span $\{x^2, 2x - x^2, x + x^3\}$.

You might need to write out the corresponding system of equations (looking at like x^k terms):

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ and solve.

(2) Does $\{x^3, x^2 - 1, x + 1\}$ span $V = P_3(\mathbb{R})$?

notice that $x^3, x^2 - 1, x + 1 \in P_3(\mathbb{R})$.

Now check if an arbitrary vector in $P_3(\mathbb{R})$ can be written as a linear combination of $x^3, x^2 - 1$ and $x + 1$. ~~That is, are~~

Let $v = ax^3 + bx^2 + cx + d \in P_3(\mathbb{R})$; $a, b, c, d \in \mathbb{R}$.

Are the always $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$ax^3 + bx^2 + cx + d = \alpha x^3 + \beta(x^2 - 1) + \gamma(x + 1) \ ?$$

Collecting like terms, we find:

$\alpha = a$

$\beta = b$

$\gamma = c$

$\gamma - \beta = d$ (or $d = c - b$).

Thus, the only polynomials from $P_3(\mathbb{R})$ in span $\{x^3, x^2 - 1, x + 1\}$ are those with $d = c - b$. That is, d cannot be arbitrarily chosen.

(3) Find two spanning sets for $V = P_3(\mathbb{R})$.

I will list several. You can check.

$X_1 = \{1, x, x^2, x^3\}$

$X_2 = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$

$X_3 = \{1, x, x^2, x^3, 3x - 7\}$

$X_4 = \{0, 1, x, x^2, x^3\}$

$X_5 = \{1 + x, x + x^2, x^2 + x^3, x^3\}$