MATH 420 – Review Concepts

1. Consider the set $\mathbb{R}^n$ and fixed vector $z \in \mathbb{R}^n$. Let vector addition be defined by $x \oplus y = x + y - z$ and scalar multiplication be defined by $a \odot x = a(x - z) + z$ for all $x, y \in \mathbb{R}^n$ and all $a \in \mathbb{R}$. in a previous homework, you were asked to show that $(\mathbb{R}^n, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

   (a) Give the meaning of the statement: $x \oplus y = x + y - z$.

   (b) Give the meaning of the statement: $a \odot x = a(x - z) + z$

   (c) Prove that the zero vector in $(\mathbb{R}^n, \oplus, \odot)$ is z.

   (d) Prove that the additive inverse of x is $2z - x$.

   (e) Prove that the multiplicative identity is 1 (an element in $\mathbb{R}$).

2. Consider a vector space $(V, \oplus, \odot)$ over $\mathbb{F}$.

   (a) Give the meaning of the statement $0 \odot x = 0$.

   (b) Prove that $0 \odot x = 0$ for all $x \in V$.

   (c) Show that the statement $0 \odot x = 0$ holds true for the vector space of Exercise 1.

3. Consider a vector space $(V, \oplus, \odot)$ over $\mathbb{F}$.

   (a) Give the meaning of the statement $-(a \odot x) = a \odot (-x)$.

   (b) Prove that $-(a \odot x) = a \odot (-x)$ for all $x \in V$ and all $a \in \mathbb{F}$.

   (c) Show that the statement $-(a \odot x) = a \odot (-x)$ holds true for the vector space of Exercise 1.

4. Consider a vector space $(V, \oplus, \odot)$ over $\mathbb{F}$. Clearly state how you would show whether or not $(W, \oplus, \odot)$ is a vector space over $\mathbb{F}$, where $W \subseteq V$.

1. Consider the set $\mathbb{R}^n$ and fixed vector $z \in \mathbb{R}^n$. Let vector addition be defined by $x \oplus y = x + y - z$ and scalar multiplication be defined by $a \odot x = a(x - z) + z$ for all $x, y \in \mathbb{R}^n$ and all $a \in \mathbb{R}$. in a previous homework, you were asked to show that $(\mathbb{R}^n, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

   (a) Give the meaning of the statement: $x \oplus y = x + y - z$.

   Vector addition of x and y is defined (using the standard operations) as summing x, y and then subtracting z.

   (b) Give the meaning of the statement: $a \odot x = a(x - z) + z$

   Scalar-vector multiplication of a with x is defined (using the standard operations) as taking the product of a with $(x - z)$ and then adding z.

   (c) Prove that the zero vector in $(\mathbb{R}^n, \oplus, \odot)$ is z.

   Proof. We show that z satisifies the properties of the zero vector in $(\mathbb{R}^n, \oplus, \odot)$, namely $x \oplus z = x$ and $z \oplus x = x$ for all $x \in \mathbb{R}^n$. Ovserve: $x \oplus z = x + z - z = x$ and $z \oplus x = z + x - z = x$. □

   (d) Prove that the additive inverse of x is $2z - x$.

   Proof. We show that $2z - x$ satisfies the properties of the additive inverse vector of x for all x in vector space $(\mathbb{R}^n, \oplus, \odot)$, namely $x \oplus (2z - x) = z$ and $(2z - x) \oplus x = z$. Observe: $x \oplus (2z - x) = x + (2z - x) - z = z$ and $(2z - x) \oplus x = (2z - x) + x - z = z$. □

   (e) Prove that the multiplicative identity is 1 (an element in $\mathbb{R}$).

   Proof. We show that the scalar 1 satisfied the properties of the multiplicative identity in $(\mathbb{R}^n, \oplus, \odot)$, namely $1 \odot x = x$ for all $x \in \mathbb{R}^n$. Observe: $1 \odot x = 1(x - z) + z = 1x - 1z + z = x$. □

2. Consider a vector space $(V, \oplus, \odot)$ over $\mathbb{F}$.

   (a) Give the meaning of the statement $0 \odot x = 0$.

   The zero scalar of the field $\mathbb{F}$ when multiplied by a vector x in the vector space V, results in the zero vector of V.

   (b) Prove that $0 \odot x = 0$ for all $x \in V$.

   Proof. We show that for arbitrary $x \in V$, $0 \odot x = 0$ by using the vector space properties in the definition of a vector space.

$$\begin{aligned}
(0 \odot x) \oplus x &= (0 \odot x) \oplus (1 \odot x) && \text{(P10)} \\
&= (0 + 1) \odot x && \text{(P7)} \\
&= 1 \odot x && \text{(scalar addition)} \\
&= x && \text{(P10)} \\
&= 0 \oplus x && \text{(P8)}
\end{aligned}$$

   Thus, by Theorem 3.4.1, $0 \odot x = 0$. □

   (c) Show that the statement $0 \odot x = 0$ holds true for the vector space of Exercise 1.

   We show that $0 \odot x = z$, where 0 is the zero scalar in $\mathbb{R}$ and z is the additive identiy vector in $(\mathbb{R}^n, \oplus, \odot)$ which is not necessarily a vector of zero values. Observe: $0 \odot x = 0(x - z) + z = z$.

3. Consider a vector space $(V, \oplus, \odot)$ over $\mathbb{F}$.

   (a) Give the meaning of the statement $-(a \odot x) = a \odot (-x)$.

   The additive inverse of the scalar product of scalar a and vector x is equal to the scalar product of a with the additive inverse of vector x.

   (b) Prove that $-(a \odot x) = a \odot (-x)$ for all $x \in V$ and all $a \in \mathbb{F}$.

   Proof. We demonstrate that the additive inverse of $a \odot x$ is equal to $a \odot (-x)$ by showing $(a \odot x) \oplus (a \odot (-x)) = 0$. Observe:

$$\begin{aligned}(a \odot x) \oplus (a \odot (-x)) &= a \odot (x \oplus (-x)) && \text{(P7)}\\ &= a \odot 0 && \text{(P9)}\\ &= 0 && \text{(Thrm 3.4.3)}\end{aligned}$$

   □

   (c) Show that the statement $-(a \odot x) = a \odot (-x)$ holds true for the vector space of Exercise 1.

   Using the fact that the additive inverse of vector x is vector $2z - x$, we find:

$$-(a \odot x) = 2z - a \odot x = 2z - a(x - z) - z = z - ax + az,$$

   and

$$a \odot (-x) = a \odot (2z - x) = a(2z - x - z) + z = z - ax + az.$$

   So, $-(a \odot x) = a \odot (-x)$.