Calculus and Optimization – Class Discussion

Consider the elevation function $f(x, y) = x^2 + 8xy + y^4 - 16$, where $f(x, y)$ is the land elevation at location (x, y).

1. Describe the geometry of $f(x, y)$. What does the function looks like in general terms?

   The Class discussion included these ideas:

   - The x^2 and y^4 terms show that the function gets larger and larger as either x or y gets large in absolute value.
   - The function can have regions where it is increasing or decreasing in certain directions.
   - The function is generally bowl shaped and more elongated in the x direction.
   - Is the function bowl shaped when not looking along the x or y axis? One way to test is to consider an arbitrary direction (ay, y) where a is any nonzero scalar. That is, consider the line where $x = ay$ and see how the function behaves in that direction.

2. Can this function have local minimizers? Local maximizers? Answer these questions without using computations.

   The Class discussion included these ideas:

   - A local minimum is a point (x, y) where the function attains its lowest value – at least in some region around (x, y).
   - In the bottom of the bowl shape there could be bumps and dips which lead to local minima and local maxima.

3. Might this function have global minimizers? Global maximizers? Answer these questions without using computations.

   The Class discussion included these ideas:

   - A global minimum is a point (x, y) where the function attains its lowest value over the entire domain.
   - The function must have a global minimum because it is bowl shaped – there must be a bottom to the bowl. (Caution: the definitive test is to find a value K such that $f(x, y) > K$ for all values of x and y.)
   - $f(x, y)$ cannot have a global maximum because no matter what (x, y) is chosen, there is always another point that has a higher function value.
   - There may be more than one global minimum. There may be infinitely many if the function is flat at the bottom.

4. What does it mean to be a stationary point of $f(x, y)$? What types of stationary points might exist?

   The Class discussion included these ideas:

   - A point x^* is a stationary point of a function $f(x)$ if $f'(x^*) = 0$.
   - A stationary point of an elevation function is a point where a ball can be placed and not roll.
   - Stationary points could be the top of a hill or the bottom of a bowl.
   - Stationary points *seem* to correspond to local maxima and minima.
   - A point (x^*, y^*) is a stationary point of a function $f(x, y)$ if

     $$\nabla f(x^*, y^*) = 0.$$

     That is, the gradient vector at (x^*, y^*) is the zero vector.
   - Stationary points could also be flat regions and saddle points.

5. Find all of the stationary points of $f(x, y)$.

   We have

   $$\nabla f(x, y) = \begin{bmatrix} 2x + 8y \\ 8x + 4y^3 \end{bmatrix},$$

   and stationary points of f are thus solutions to:

   $$\begin{aligned} 2x + 8y &= 0 \\ 8x + 4y^3 &= 0 \end{aligned}$$

   Solving for x in the first equation and using substitution yields the condition on y: $4y(y^2 - 8) = 0$). Thus, there are three stationary points: $p_1 = (0, 0)$, $p_2 = (-8\sqrt{2}, 2\sqrt{2})$ and $p_3 = (8\sqrt{2}, -2\sqrt{2})$.

6. How can we determine if a given stationary point is a local minimum, a local maximum or neither?

   The Class discussion included these ideas:

   - For a function $f(x)$, with stationary point x^*, we know that x^* is a local maximum if $f''(x^*) < 0$ and x^* is a local minimum if $f''(x^*) > 0$. If $f'(x^*) = 0$ then we have no information – x^* could be a local maximum, a local minimum, or any place where the function flattens out.
   - Try $f(x) = x^3$ and $f(x) = x^4$ to test the above idea and see what happens.
   - A stationary point (x^*, y^*) of a function $f(x, y)$ is a local minimum (maximum) if all of the eigenvalues of the Hessian matrix $\nabla^2 f(x^*, y^*)$ are negative (positive).
   - The Hessian matrix is

     $$\nabla^2 f(x, y) = \begin{bmatrix} \frac{\partial^2}{\partial x^2} f(x, y) & \frac{\partial^2}{\partial x \partial y} f(x, y) \\ \\ \frac{\partial^2}{\partial y \partial x} f(x, y) & \frac{\partial^2}{\partial y^2} f(x, y) \end{bmatrix}.$$

- If the eigenvalues of the Hessian matrix are of mixed signs, then the point (x^*, y^*) is a saddle point.
- if any eigenvalue of the Hessian matrix is zero, then we cannot make firm conclusions about whether the point in question is a local minimum or local maximum.

7. Characterize all of the stationary points of $f(x, y)$.

   Homework.