Principles of Optimization
Required Problems

Below is the list of Required Problems. Some of these come out of your textbook, "The Basics of Practical Optimization," by Adam B. Levy. All reference numbers are referring to this book.

1. For a factory that manufactures trucks and cars, each vehicle must go through an assembly stage and a painting stage. If the painting was entirely devoted to cars, 500 per day could be painted. On the other hand, 400 trucks per day could be painted per day if the painting was devoted to trucks. The analagous numbers for assembly are 80 cars per day and 65 trucks per day. Each car produces \$1300 profit and each truck produces \$1500 profit.

   Write and solve an optimization model to answer the following question: How many cars and trucks should the factory produce to maximize profit?

2. A person's daily nutritional requirements include the need for 3 units of vitamin A and 1.5 units of vitamin C. Bananas sell for \$6 per pound, and each pound contains 5 units of vitamin A and 1 unit of vitamin C. Rice sells for \$1 per pound, and each pound contains 1 unit of each vitamin.

   (a) Write and solve an optimization model to answer the following question: With only bananas and rice to eat, what is the least expensive scheme that satisfies the person's requirements?

   (b) Write and solve an optimization model to answer the following question: What is the least expensive scheme that satisfies the person's requirements exactly?

3. A supermarket can be staffed with full- and part-time employees in any way that covers the labor requirements of 400 total person hours on Saturday, 400 on Sunday, 200 on Monday, 200 on Tuesday, 250 on Wednesday, 250 on Thursday, and 300 on Friday. Union rules require that each full-time employee work 5 consecutive full days (8 hours each) followed by 2 days off, but the supermarket can require them to work 1 day of overtime each week (with only 1 day off). A full-time employee costs the supermarket \$10 per hour and a part-time employee costs \$8 per hour.

   Write and solve an optimization model to answer the following question: If each employee is paid \$80 a day for each of their first 5 days and \$90 for the overtime day (if any), how can the supermarket meet its daily labor requirements at least cost?

4. An oil company has 6,000 barrels of oil 1 (with quality level 10) and 11,000 barrels of oil 2 (with quality level 6) and cannot restock for this selling period. The company sells two products made from a combination of the two oils: gasoline, which must have an average quality level of at least 8, and heating oil, which must have an average quality level of at least 7. Each dollar spent advertising gasoline results in 3 barrels of demand and each dollar spent advertising heating oil results in 7 barrels of demand, and there is no demand without advertising.

Write and solve an optimization model to answer the following question: If gasoline is sold for $35 per barrel and heating oil is sold for $30 per barrel, what should the oil company sell to maximize profit?

5. A violin maker expects the following demands for violins: 5 violins in the first quarter, 4 violins in the second quarter, 4 violins in the third quarter, 3 violins in the fourth quarter. There is an inventory of 6 violins at the beginning of the first quarter. During each quarter, the violin maker can produce 2 violins at a cost of $200 per violin, and can use overtime employees to make additional violins if necessary, but at a cost of $300 per violin.

   Write and solve an optimization model to answer the following question: If, at the end of each quarter, a holding cost of $25 per violin is incurred for violins not sold, how should the violin maker schedule production in order to minimize the sum of the production and inventory costs during the four quarters while meeting the expected demand?

6. An automaker advertises with commercial spots on three types of TV programs: dramas (costing $150,000 per minute and seen by 8 million women and 4 million men), sports (costing $180,000 per minute and seen by 2 million women and 12 million men), and news (costing $130,000 per minute and seen by 4 million women and 5 million men).

   (a) Write and solve an optimization model giving a strategy to reach at least 35 million women and 35 million men at least cost.

   (b) Now do the same under the further stipulation that only one-minute-long commercial spots can be used.

7. A prospective gold mine has been subdivided into 25 5-ton blocks in the following formation:

Level 1	1	2	3	4	5	6	7	8	9	$200
Level 2		10	11	12	13	14	15	16		$300
Level 3			17	18	19	20	21			$400
Level 4				22	23	24				$500
Level 5					25					$600

   where the extraction costs per ton at each level are given in the final column. The only exception is block 9, which sits under a wetland and as a result requires an additional $3,650 per ton to extract. The extraction of any block at a lower level requires the extraction of the three nearest blocks above it (so, for example, blocks 11, 12, and 13 must be extracted before block 18 can be extracted). The market values in $/ton of each block are as follows:

Level 1	100	300	200	20	10	10	100	10	50
Level 2		400	500	300	200	100	100	150	
Level 3			400	300	1000	1500	500		
Level 4				2000	2100	1900			
Level 5					3000				

Write and solve an optimization model to determine which blocks should be extracted to maximize profit. What is the result if the additional per-ton cost of extracting block 9 is $3350 instead of $3650?

8. Five fast-food workers are to be assigned to five different stations, the table below gives the productivity on a scale of 1-10 of each worker at each station.

Worker	Fries	Burgers	Ice Cream	Register	Drive-thru
1	1	6	6	5	9
2	3	6	9	4	10
3	2	5	6	3	6
4	3	6	6	2	7
5	4	7	7	2	8

Write and solve an optimization model to answer the following question: If the total productivity of the restaurant is the sum of the productivities of the workers assigned to the stations, what assignment of workers to stations maximizes total productivity?

9. An online DVD rental company would like to measure the efficiency of five of its distribution centers in New England: one each in Maine, Vermont, and New Hampshire, and two in Massachusetts. This is done via "data envelopment analysis," which compares ratios of weighted sums of outputs for a center to weighted sums of inputs for the same center. The efficiency of any distribution center is defined to be the maximum ratio over all choices of (nonnegative) weights for which none of the corresponding five rations (one for each center) exceeds 1. The table below shows the average daily inputs and outputs of each distribution center.

Inputs	ME	VT	NH	MA 1	MA 2
Value of DVD stock (in $1000's)	1200	1100	1300	1500	1400
Wages	$2,000	$1,580	$2,100	$3,800	$4,000
Outputs	**ME**	**VT**	**NH**	**MA 1**	**MA2**
Deliveries	1100	950	1120	1220	1190
Restock requests	100	70	150	210	230

Write and solve five optimization models to find the efficiency of each of the five distribution centers.

10. A wooden puzzle manufacturer uses four main processes to produce its finished puzzles: wood preparation, painting, cutting, and sanding. The manufacturer uses the same four processing stations to make five different puzzles, whose respective times for each state of processing are given in the table below. Once a puzzle has entered the sequence of processing stations, it cannot be "passed" by a puzzle which entered the sequence later, and puzzles can wait between stations.

Puzzle Type	1	2	3	4	5
Wood preparation	4	3	2	3	3
Painting	7	8	6	9	7
Cutting	5	5	6	4	4
Sanding	3	2	3	1	3

Write and solve an optimization model to find the sequence of processing which minimizes the total time to complete exactly one of each type of puzzle.

11. The state of Washington plans to build pollution removal facilities along the Columbia River. Three sites are under consideration. The cost of building at each site and the cost of treating each ton of water is given in the table. The legislature requires that at least 80,000 tons of pollutant 1 and 50,000 tons of pollutant 2 be removed from the river. The amounts of each pollutant removed from treated water is given in the table. Write and solve an optimization model to find a lowest-cost strategy that meets the legislatures goals.

site	cost of building ($)	cost of treating 1 ton of water ($)	tons pollutant 1 per ton of treated water	tons pollutant 2 per ton of treated water
1	100,000	20	0.40	0.30
2	60,000	30	0.25	0.20
3	40,000	40	0.20	0.25

12. You are considering which of ten acquaintances to invite to a party: Abdul, Baji, Carlos, Daniel, Esther, Fouzia, Goro, Hemene, Ingrid and Jie. Write and solve an optimization model that maximizes the number of guests under the following conditions.

    (a) At least two women (Baji, Esther, Fouzia, Ingrid, Jie) and at least two men (Abdul, Carlos, Daniel, Goro, Hemene) should be invited.

    (b) Fouzia and Daniel should not both be invited.

    (c) If Abdul is invited then Jie should also be invited.

    (d) If Baji and Daniel are invited then Hemene should also be invited.

    (e) If Esther and Jie are both invited then neither Goro nor Abdul should be invited.

13. A breakfast cereal manufacturer wishes to make a new product from some combination of the five ingredients: oats, wheat, flax, honey and blueberries. The company machinery dispenses these ingredients in unit quantities into a large mixing vat. Oats, wheat and berries are dispensed in units of cups; honey and flax are dispensed in units of tablespoons. The table shows the calories contained in one unit of each ingredient. Write and solve an optimization model to find the mixture of ingredients that most closely provides a mixture of 2500 calories under the additional constraints that (a) at least on unit of each ingredient must be used, (b) no more than 7 units of any ingredient can be used and (c) the total volume of ingredients cannot exceed 9 cups.

	oats	honey	wheat	flax	blueberries
unit	1 cup	1 tbsp	1 cup	1 tbsp	1 cup
calories	158	64	651	55	85

14. A farmer is planning an orchard of mixed apple, pear and cherry trees that can hold a maximum of 350 trees. The seasonal cost of labor per tree is $150, $200 and $250,

respectively. The seasonal cost of materials per tree is \$275, \$175 and \$125, respectively. Planting logistics require that if any apple trees are planted then at least 150 must be planted. Similar limits on pear and cherry trees are 50 and 80, respectively. Write and solve an optimization model that maximizes the number of trees which the farmer can plant when labor costs are limited to a maximum of \$50,000 and materials costs are limited to a maximum of \$60,000.

15. Complete the following:

    (a) Sketch the three surfaces of revolution corresponding to the three surface-generating functions shown in Figure 2.4.

    (b) Give the equations describing $x(t)$ for two surface-generating functions, and compute the value of Newton's integral for each.

16. Complete the following:

    (a) Explain how the difference quotient approximations in equation 2.6 are consistent with Figure 2.5.

    (b) Without solving the discretization problem of minimizing Newton(x) defined in (2.7) over all values of x, explain why its solution can't be the solution to the original integral optimization problem whose graph appears in Figure 2.6.

    (c) Give the discretization problem that results from using left endpoints instead of right endpoints.

    (d) Give the stationary point equation for the corresponding "left-endpoint" discretization function.

17. Suppose we have N data points in $\mathbb{R}^2$, $\{(x_k, y_k)\}_{k=1}^N$ and a model function

$$y = f(x) = a_0 + a_1 \sin(x) + a_2 e^{-x/7} + a_3 x^2$$

which we believe should accurately fit our data for some choice of parameter values $a = (a_0, a_1, a_2, a_3)$. We can formulate an optimization model for choosing optimal parameter values:

$$\min_a \sum_{k=1}^{N} \left| y_k - \left(a_0 + a_1 \sin(x_k) + a_2 e^{-x_k/7} + a_3 x_k^2 \right) \right| .$$

    (a) Carefully describe what this objective function measures.

    (b) Reformulate this optimization problem to be a linear program.

18. Implement Newton's optimization method for

    (a) the two-variable discretization function (3.4) and

    (b) the monthly payment function (2.3).

19. Complete the following:

(a) Write the update (3.9) for the case of the two-variable discretization function (3.4) with $(x_{\text{old}}, y_{\text{old}}) = (1, 2)$.

(b) What does the update (3.9) look like when the function f has only one variable x?

(c) Give an example of a Taylor quadratic that does not have a stationary point.

20. Implement the steepest descent method with an exact line-search for minimizing

    (a) the two-variable discretization function (3.4) and

    (b) the monthly payment function (2.3),

    and use a contour plot to chart the progress in each case.

21. Consider the following two functions:

$$f_1(x, y) = x^2 + y^2 \text{ and } f_2(x, y) = x^2 + \frac{y^2}{2}.$$

    (a) Use the contour plots in Figure 4.2 to illustrate one step of the steepest descent method (4.1) applied to each function from the current guess $(x_{\text{old}}, y_{\text{old}}) = (1, 1)$, using each of the following three line-search alternatives:

    - No line-search, so $\alpha = 1$.
    - The backtracking line-search.
    - The exact line-search.

    (b) What is going to happen in each case as more steps are taken? What is significant about the function f_1 that causes the behaviour you observe?

    (c) Now consider a general function f. It is a fact that consecutive steps of the steepest descent method with an exact line-search are perpendicular.

    i. Verify this fact conceptually by explaining the relationship between the line of search and the contour of f at the new guess.

    ii. Prove this fact analytically by applying the two-variable chain rule to the derivative with respect to α of the function being minimized in the α-optimization problem (4.2).

22. Consider the one-variable function

$$f(x) = \log(1 + e^{2x}) - x.$$

    (a) Find the first and second derivatives, and use them to verify that there is a minimizer at $x = 0$.

    (b) Apply Newton's optimization method to minimize this function starting with guess $x_{\text{old}} = 1$.

    (c) Do the same thing but starting with guess $x_{\text{old}} = 1.1$.

(d) Verify that the second derivative is always positive (this is the one-variable analogue of a positive-definite second-derivative matrix) but that it gets very small at the ends of the x-axis.

23. Complete the following:

    (a) If relationship (5.3) is viewed as a linear relationship instead, what would be the "constant" multiplying the old error on the right site? Use this to explain why having a relationship such as (5.3) persist in the limit, regardless of the size of δ, is at least as good as a superlinear order of convergence.

    (b) Generate a sequence of guesses that appears to approach $x^* = 1$ and that exhibits quadratic order of convergence with constant $\delta = 4$.

24. Track the error fraction on the left side of (5.6) for Newton's optimization method you used in Required Problem 18 (above) on

    (a) the two-variable discretization function (3.4) and

    (b) the monthly payment function (2.3).

    You should observe a quadratic order of convergence in each case. What are the corresponding constants δ?

25. Modify the steepest descent method with an exact line-search from Required Problem 20 by computing the errors at each step. Show that there is a linear order of convergence for

    (a) the two-variable discretization function (3.4) and

    (b) the monthly payment function (2.3),

    and give the constant fractions δ in each case.

26. Complete the following:

    (a) Which of the surface-generating functions $x(t)$ whose graphs are pictured in Figure 2.4 satisfy the constraint $x'(t) \geq 0$?

    (b) What does the constraint $x'(t) \geq 0$ say in general about a surface-generating function?

    (c) Give the single-variable function that results from a discretization of the integral (6.1) using t-values $0, 1/2$, and 1.

    (d) How does the constraint $x'(t) \geq 0$ on the surface-generating function translate into constraints on the single variable in this discretization?

27. Again consider the discretization using t-values $0, 1/2$, and 1 of Newton's aerodynamics problem in rare media from Required Problem 26. Recall that the constraint $x'(t) \geq 0$ in this case translates into a simple interval constraint on the single variable of the discretization problem.

(a) Using general factors $\lambda_1 > 0$ and $\lambda_2 > 0$, create an appropriate twice-differentiable penalty function p_3 for the interval in this case.

(b) Explore the effect on the penalized minimizers of different choices of factors until you can answer the following questions:

i. Is there any combination of factors for which the penalized minimizer solves the original discretization problem?

ii. Do changes in each of the factors λ_1 and λ_2 have similar or very different effects on the location of the corresponding penalized minimizers?

iii. For which values of factors λ_1 and λ_2 does the penalty function approach the ideal penalty function for the interval constraint in this case?

iv. Does this approach happen graphically from the interior or the exterior of the interval constraint?

28. Use the reduction method to solve the inequality constrained optimization problem resulting from a discretization using t values $0, 1/4, 1/2, 3/4$, and 1 of Newton's aerodynamics problem in rare media (Required problem 26): Find the surface-generating function $x(t)$ (passing through $(0,0)$ and $(1,1)$) satisfying $x'(t) \geq 0$ and minimizing the integral

$$\int_0^1 \frac{t}{1+(x'(t))^2}\, dt.$$

29. Apply Newton's optimization method to the Lagrangian in order to solve the problem

$$\min_{x,y\in\mathbb{R}} x^2 - 2xy + 2y^2$$
$$\text{s.t. } y = (x-y)^2 + 1,$$

and give a full convergence analysis in this case.

30. Implement the SQP method for the problem

$$\min_{x,y\in\mathbb{R}} x^2 - 2xy + 2y^2$$
$$\text{s.t } y \geq (x-y)^2 + 1.$$

Give a full convergence analysis and compare your results to those from Required Problem 29.