Math 364 – Examples of Binary Variable Constraints

Suppose we have a set of binary variables x_k, $k = 1, 2, \cdots, n$. The following are examples of typical optimization problem conditions and equivalent linear constraint sets.

1. It must be between M and N ($\leq M$) of the x_k which equal 1.

$$N \leq \sum_{k=1}^{n} x_k \leq M$$

2. x_1, x_2 and x_3 cannot all have the same value.

$$1 \leq x_1 + x_2 + x_3 \leq 2$$

3. Only one of the x_k can equal 1, unless $x_1 = 1$, then one additional x_k can also equal 1.

$$\sum_{k=2}^{n} x_k \leq 1$$

4. if $x_k = 1$ then $x_{j>k} = 0$.

$$\sum_{k=1}^{n} x_k \leq 1$$

5. If ($x_1 = 1$ and $x_2 = 1$) then $x_3 = 0$.

$$x_1 + x_2 + x_3 \leq 2$$

6. If $x_1 + x_2 \geq 1$ then $x_3 = 1$.

$$x_1 + x_2 \leq 2x_3$$

7. If $x_1 = 1$ then $x_2 = x_3 = 0$.

$$2x_1 + x_2 + x_3 \leq 2$$

8. If $x_1 = x_2 = 1$ then $x_3 = x_4 = 0$.

$$2\,(x_1 + x_2) + (x_3 + x_4) \leq 4$$

9. It must be an even number of the x_k which equal 1.

$$\sum_{k=1}^{n} x_k = 2y$$
$$y \in \mathbb{Z}$$

10. It must be an odd number of the x_k which equal 1.

$$\sum_{k=1}^{n} x_k = 2y + 1$$
$$y \in \mathbb{Z}$$

11. x_3 can only equal zero if $x_1 + x_2 = 1$.

$$x_1 + x_2 + x_3 \geq 1$$
$$x_1 + x_2 - x_3 \leq 1$$

12. If $x_1 + x_2 + x_3 = 2$ then $x_4 = 0$.

$$x_1 + x_2 + x_3 + x_4 \leq 2 \text{ OR } x_1 + x_2 + x_3 \geq 3$$

which is equivalent to

$$
\begin{aligned}
&x_1 + x_2 + x_3 + x_4 \leq 2 + 2y \\
&x_1 + x_2 + x_3 \geq 3y \\
&y \in \{0, 1\}
\end{aligned}
$$

13. $x_3 = x_1 \wedge x_2$.

$$
\begin{aligned}
&x_3 \geq x_1 + x_2 - 1 \\
&x_3 \leq x_1 \\
&x_3 \leq x_2
\end{aligned}
$$

14. $x_3 = x_1 \vee x_2$.

$$
\begin{aligned}
&x_3 \leq x_1 + x_2 \\
&x_3 \geq x_1 \\
&x_3 \geq x_2
\end{aligned}
$$

15. $x_3 = x_1 \wedge \neg x_2$.

$$
\begin{aligned}
&x_3 \geq x_1 - x_2 \\
&x_3 \leq x_1 \\
&x_3 \leq 1 - x_2
\end{aligned}
$$

16. $x_3 = x_1 \oplus x_2$, where $\oplus$ means "exclusive or."

$$
\begin{aligned}
&x_3 \leq x_1 + x_2 \\
&x_3 \leq 2 - x_1 - x_2 \\
&x_3 \geq x_1 - x_2 \\
&x_3 \geq x_2 - x_1
\end{aligned}
$$