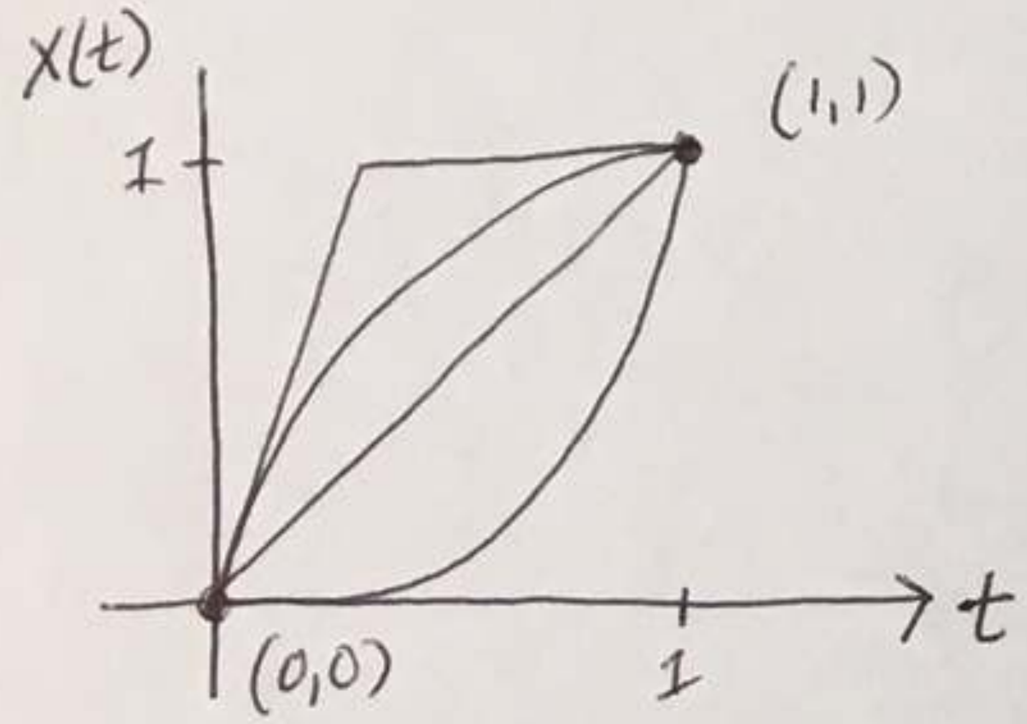
More on Penalty Functions for Constrained Optimization

1. Newton found that the air resistance of a surface of revolution, defined by function $x(t)$, in a "**rare**" medium is proportional to the integral

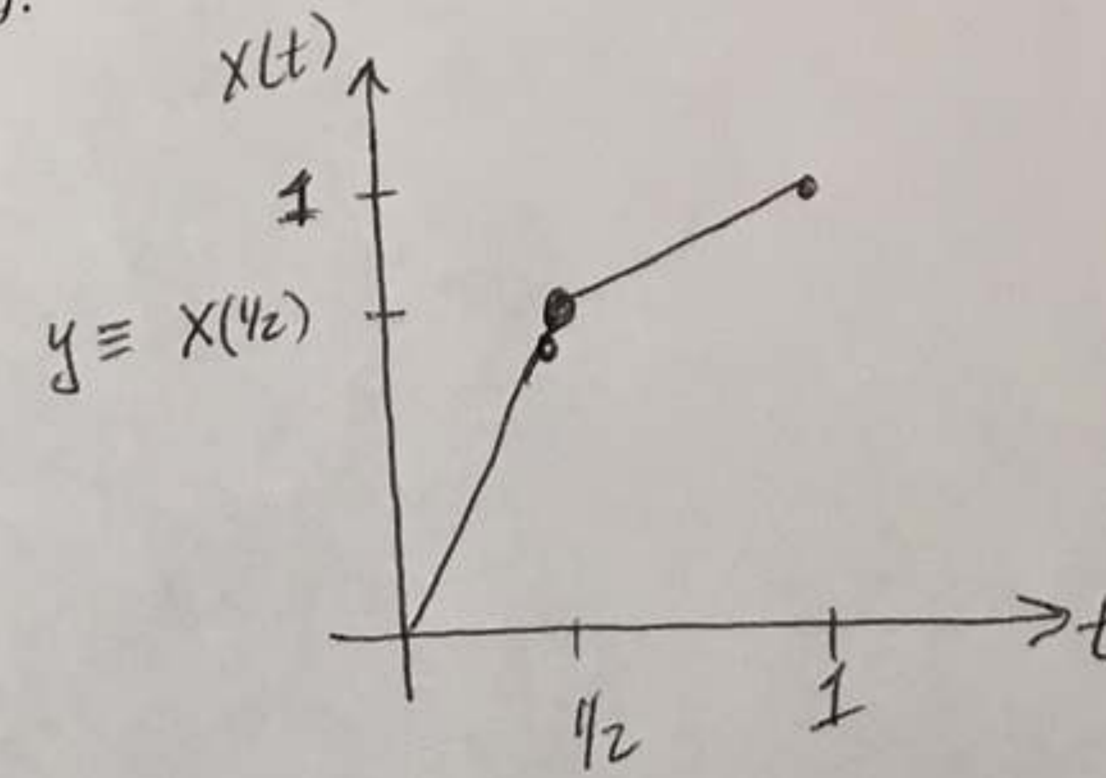
$$\int_0^1 \frac{t\,dt}{1+[x'(t)]^2},$$

where, $x(0)=0$, $x(1)=1$, $x'(t)\geq 0$. Sketch several surface-generating functions $x(t)$ that satisfy Newton's conditions. What does the constraint $x'(t)\geq 0$ say in general about appropriate surface-generating functions?



Each surface generating function is non-decreasing in t.
(Look up "monotonic functions")

3. Construct the single-variable function $f(y)$ that results from the left end point discretization of the integral for $t = 0, 1/2$, and 1. Carefully define y.



$$f(y) = \frac{(0)(\frac{1}{2}-0)}{1+\left(\frac{y-0}{\frac{1}{2}-0}\right)^2} + \frac{(\frac{1}{2})(1-\frac{1}{2})}{1+\left(\frac{1-y}{1-\frac{1}{2}}\right)^2}$$

$$f(y) = \frac{1}{4+16(1-y)^2}$$

4. How does the constraint $x'(t) \geq 0$ translate into constraints on the new variable y? Solve this constrained optimization problem. That is, find y^* that produces the curve whose surface of revolution has minimum air resistance. Discuss your result.

For $x'(t) \geq 0$ we must have $0 \leq y \leq 1$, That is, $x(0) \leq x(1/2) \leq x(1)$.

The optimization problem is

$$\boxed{\begin{array}{ll} \min & f(y) = \dfrac{1}{4+16(1-y)^2} \\ \text{s.t.} & 0 \leq y \leq 1 \end{array}}$$

First consider all stationary points of $f(y)$.

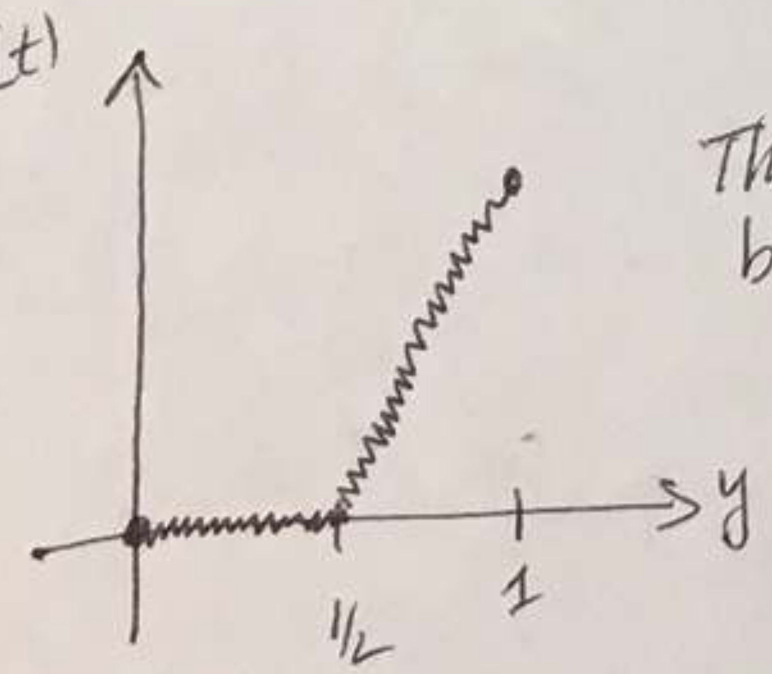
$$f'(y) = \frac{-2(1-y)}{(4+16(1-y)^2)^2} \overset{\text{set}}{=} 0 \Rightarrow y = 1$$

we see that for any $y \neq 1$, $f(y) < \frac{1}{4}$, so $y=1$ is a local maximizer.

Since there are no local minima, the solution must lie on the boundary of the feasible region: either $y=0$ or $y=1$

$f(0) = \frac{1}{20}$, $f(1) = \frac{1}{4}$. Thus, the solution $\boxed{y^* = 0 \quad f(y^*) = \frac{1}{20}}$

The solution is:

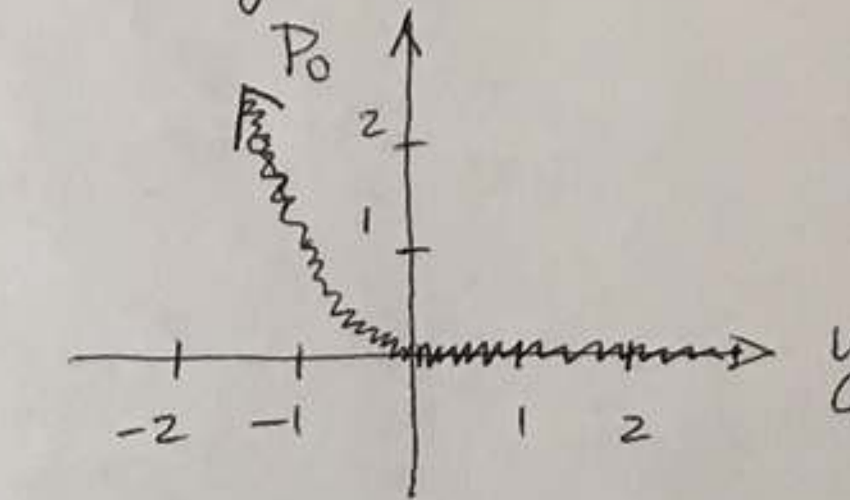


This function corresponds to a blunt tip surface of revolution.

5. Show that $p_0(y) = -\min\{0, y\}^3$ is a twice continuously differentiable penalty function that approximates the constraint $y \geq 0$. Justify!

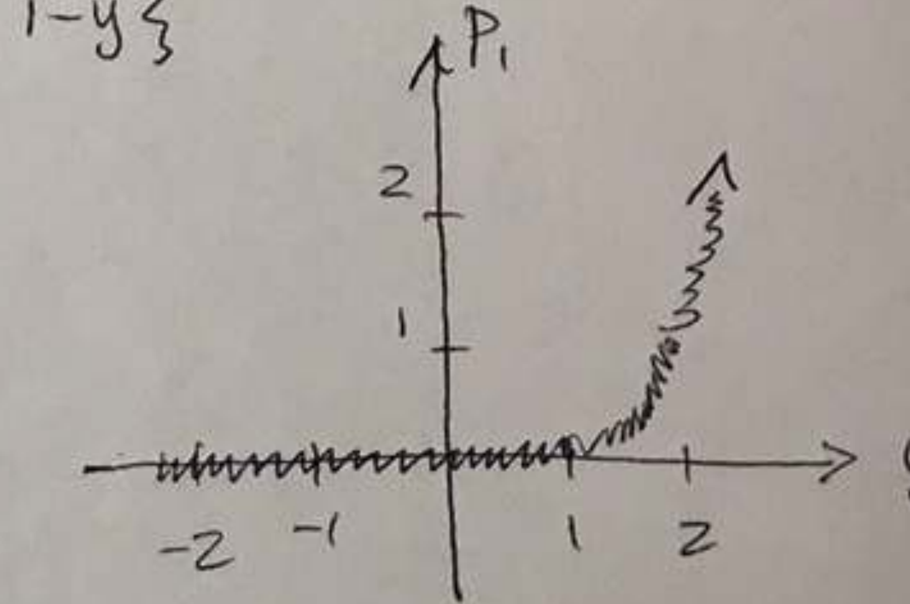
$$P_0(y) = \begin{cases} 0 & \text{if } y \geq 0 \\ -y^3 & \text{if } y < 0 \end{cases} \quad P_0'(y) = \begin{cases} 0 & \text{if } y \geq 0 \\ -3y^2 & \text{if } y < 0 \end{cases} \quad P_0''(y) = \begin{cases} 0 & \text{if } y \geq 0 \\ -6y & \text{if } y < 0 \end{cases}$$

P_0, P_0', P_0'' are continuous. $P_0(y)$ works as a penalty function associated with $y \geq 0$ because it does not penalize $y \geq 0$ but provides a "large" contribution to the objective function when $y < 0$.



6. Formulate a similar penalty function that approximates the constraint $y \leq 1$.

$P_1(y) \equiv -\min\{0, 1-y\}^3$



7. Using parameters $\lambda_1 > 0$ and $\lambda_2 > 0$, create an unconstrained optimization problem whose solution approximates the discretized rare medium optimization problem. (Further analysis of this example is the subject of one of your required homework problems.)

$$\boxed{\min \quad \frac{1}{4+16(1-y)^2} + \lambda_1 P_0(y) + \lambda_2 P_1(y)}$$