Math 364 – Homework #2 Solutions

Due: Tuesday September 5, in class

1. Briefly explain why the given optimization problems are not linear programs.

   (a)

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & f(x) = (x_1-2)^2 + (x_2-1)^2 \\ \text{s.t.} \quad & x_1 + x_2 \le 0 \end{aligned}$$

   (b)

$$\begin{aligned} \min_{x\in\mathbb{Z}^2} \quad & f(x) = 10x_1 - 33x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 0 \\ & x_1 + 5x_2 = 7 \end{aligned}$$

   (c)

$$\begin{aligned} \max_{x\in\mathbb{R}^2} \quad & f(x) = 2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1(1+x_2) \le 5 \\ & 2x_1 + 3x_2 \ge -4 \\ & x_1 - x_2 = 0 \end{aligned}$$

2. Show, by using sketches of several representative Linear Programs in two dimensions, that if an optimal solution exists for a LP, then at least one vertex of the feasible region is optimal. Sketches should include feasible regions and level curves of objective funcions. Provide a convincing argument. In words, provide a justification that the same conclusion is true in higher dimensions.

3. Graphically, we can solve a linear program by checking each vertex of the feasible region. Provide a similar solution method for linear programs where we cannot determine all of the vertices directly. (This can occur if the problem is high-dimensional and cannot be drawn, or has many constraints and would be too time consuming to sketch.)

1 (a)
$$\min_{x\in\mathbb{R}^2} f(x) = (x_1-2)^2+(x_2-1)^2$$
$$\text{s.t. } x_1+x_2 \le 0$$

This optimization problem is not a linear program because the objective is nonlinear. In particular, $f(x)$ cannot be written $f(x) = c^T x$ for some constant vector c.

(b)
$$\min_{x\in\mathbb{Z}^2} f(x) = 10x_1 - 33x_2$$
$$\text{s.t. } x_1+x_2 \le 0$$
$$x_1+5x_2 = 7$$

This optimization problem is not a linear program because the decision variables are restricted to the set of integers. This is an integer program.

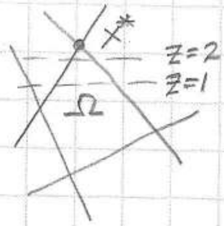
(c)
$$\max_{x\in\mathbb{R}^2} f(x) = 2x_1 - 3x_2$$
$$\text{s.t. } x_1(1-x_2) \le 5$$
$$2x_1+3x_2 \ge -4$$
$$x_1 - x_2 = 0$$

This optimization problem is not a linear program because it contains a nonlinear constraint. Namely, $x_1(1-x_2) \le 5$ cannot be written $g(x) = a^T x \le 5$ for some constant vector a.

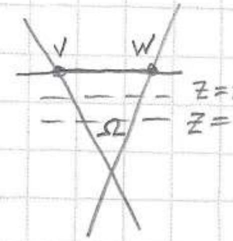
2 Consider the 2D LP representations on the next page. The first 3 have bounded feasible regions, the last four have unbounded feasible regions. Without loss of generality, assume maximization with objective value increasing vertically. Scenarios (a) and (d) have no solution. Scenarios (b) and (f) have unique solutions, labeled x^*. Scenarios (c) and (g) have infinitely many solutions along the line segment connecting points v and w. Scenario (e) has infinitely many solutions along the ray extending from point v to the right.

(a) $\Omega = \phi$

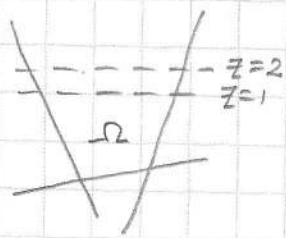
(b)



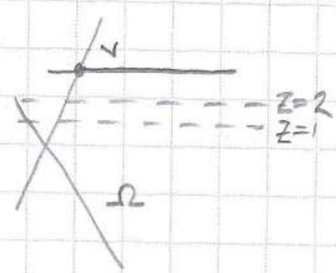
(c)



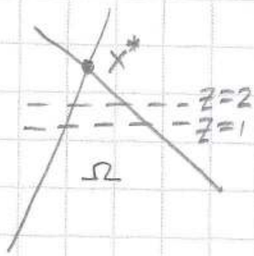
(d)



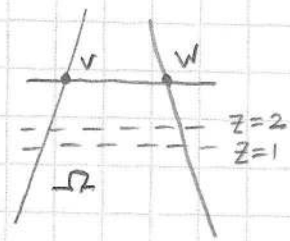
(e)



(f)



(g)



Clearly, if a 2D LP has an optimal solution, at least one optimal solution is a vertex of the feasible region Ω. In the above scenarios, such vertex optimal solutions are shown as points x^*, v and w.

In higher dimensions, we see that if an optimal solution exists then the set of optimal solutions must be a vertex, edge, face, etc. of the polytope feasible region. In each case, such a set contains vertices.

$\boxed{3}$ In turn, consider all combinations of n constraint boundaries. For each determine if the system of linear equations has a unique solution; If so check feasibility; If so, compute the objective value. Among all objective values, choose the best. The corresponding point is the candidate optimal vertex. If the problem is bounded then it is optimal. If no objective values were computed (no potentially optimal vertices were found) then the problem is infeasible.