MATH 364 HW #1 Solutions

For problems 1-5 we follow the general procedure of finding solutions as either second order stationary points ($\nabla f(x) = 0$ and either $\nabla^2 f(x) \leq 0$ or $\nabla^2 f(x) \geq 0$) or optimal points on the boundary of the feasible region. We must be careful to consider objective function values in unbounded feasible directions.

1 $\min_{x \in \mathbb{R}^2} f(x) = 2x_1^2 + 4x_2^2 + 3 + 2x_1 x_2$

$$\nabla f(x) = \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_1 + 8x_2 \end{bmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}$$

If we set $\nabla f(x) = 0$ we find the single stationary point $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. At this point (and any other point in this case) the eigenvalues of $\nabla^2 f(x)$ are $\lambda = 5 \pm \sqrt{13}$. Since $\nabla^2 f(x) > 0$, $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a second order local minimizer. We must check the behavior of $f(x)$ is unbounded directions. The problem is unconstrained so we consider arbitrary direction $w = \begin{bmatrix} u \\ v \end{bmatrix}$ and scalar β.

$$\begin{aligned} \lim_{\beta \to \infty} f(\beta w) &= \lim_{\beta \to \infty} 2\beta^2 u^2 + 2\beta^2 uv + 4\beta^2 v^2 + 3 \\ &= \lim_{\beta \to \infty} \beta(u+v)^2 + \beta u^2 + 3\beta v^2 + 3 \\ &= \lim_{\beta \to \infty} \beta\left[(u+v)^2 + u^2 + 3v^2\right] + 3 \\ &= \infty \end{aligned}$$

Thus, smaller objective values do not occur in unbounded directions.

Solution: $\boxed{x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad z^* = f(x^*) = 3}$

2) $\max_{x \in \mathbb{R}^2} f(x) = 2x_1^2 + 4x_2^2 + 3 + 2x_1x_2$

We see from problem #1 that there are no local maximizers and the objective function is not bounded above. Thus, this problem is unbounded and no solution exists.

3) $\min_{x \in \mathbb{R}^2} f(x) = (x_1-2)^2 + (x_2-1)^2$

s.t. $x_1 + x_2 \leq 0$

$$\nabla f(x) = \begin{bmatrix} 2(x_1-2) \\ 2(x_2-1) \end{bmatrix} \qquad \nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

If we set $\nabla f(x) = 0$ we find the only stationary point $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. However, this point is infeasible, not satisfying the constraint. Thus, any minimizer must be on the boundary of the feasible region, where $x_1 + x_2 = 0$. Furthermore, the objective is unbounded similarly to problem #1. Checking the boundary, we let $x_2 = -x_1$ and use the same procedure:

$f(x_1) = (x_1-2)^2 + (-x_1-1)^2 = 2x_1^2 - 2x_1 + 5.$

$f'(x_1) = 4x_1 - 2, \quad f''(x_1) = 4.$

Setting $f'(x_1) = 0$ and noting that $f''(x_1) > 0$ we find that the minimizer occurs at $x_1 = 1/2$, $x_2 = -1/2$.

Thus, the solution

$$\boxed{x^* = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \quad z^* = f(x^*) = 9/2}$$

4 $\min_{x \in \mathbb{R}^2} f(x) = (x_1-2)^2 + x_2^2$

s.t. $\sin(x_1) \geq 2$

$x_1 + x_2 \geq 2$

Note that the feasible region is empty because $\sin(x_1) < 2$ for all $x_1 \in \mathbb{R}$. The problem is infeasible.

5 $\min_{x \in \mathbb{R}^2} f(x) = \frac{1}{2}(x_1^2+x_2^2) + e^{-(x_1^2+x_2^2)}$

$$\nabla f(x) = \begin{bmatrix} x_1 - 2x_1 e^{-(x_1^2+x_2^2)} \\ x_2 - 2x_2 e^{-(x_1^2+x_2^2)} \end{bmatrix} = \begin{bmatrix} x_1(1-2e^{-r^2}) \\ x_2(1-2e^{-r^2}) \end{bmatrix}, \; r^2 = x_1^2+x_2^2$$

$$\nabla^2 f(x) = \begin{bmatrix} 1-2e^{-r^2}+4x_1^2e^{-r^2} & +4x_1x_2e^{-r^2} \\ +4x_1x_2e^{-r^2} & 1-2e^{-r^2}+4x_2^2e^{-r^2} \end{bmatrix}$$

One stationary point occurs at $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ where $\nabla^2 f(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

This point is a local maximum because $\nabla^2 f(x) < 0$. The other set of stationary points occurs at $1-2e^{-r^2} = 0$ or equivalently, $x_1^2 + x_2^2 = \ln 2$. This set of points lies on a circle of radius $\sqrt{\ln 2}$ centered at the origin. Similarly, to problem #1, $f(x)$ is not bounded above in any direction, so the set of points satisfying $x_1^2 + x_2^2 = \ln 2$ is the only remaining set of possible minimizers. At these points we obtain the Hessian matrix

$$\nabla^2 f(x) = \begin{bmatrix} 2x_1^2 & 2x_1x_2 \\ 2x_1x_2 & 2x_2^2 \end{bmatrix} \quad \text{with eigenvalves } \lambda = 0, 2\sqrt{\ln z}.$$

One eigenvalve is positive, the other is zero. So this test is inconclusive. I leave it to the student to show that the set $\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = \ln z\}$ is in fact the set of minimizers. Hint: reduce the problem to a 1-dimensional problem in variable r.

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