Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shall show in Chap. 18.

PROBLEM SET 13.4

1–10 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? [Use (1) or (7).]

1. $f(z) = z^4$ **2.** $f(z) = \text{Im}\,(z^2)$

3. $e^{2x}(\cos y + i \sin y)$ **4.** $f(z) = 1/(1 - z^4)$

5. $e^{-x}(\cos y - i \sin y)$ **6.** $f(z) = \text{Arg}\,\pi z$

7. $f(z) = \text{Re}\, z + \text{Im}\, z$ **8.** $f(z) = \ln |z| + i\,\text{Arg}\, z$

9. $f(z) = i/z^8$ **10.** $f(z) = z^2 + 1/z^2$

11. (Cauchy–Riemann equations in polar form) Derive (7) from (1).

12–21 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = xy$ **13.** $v = xy$

14. $v = -y/(x^2 + y^2)$ **15.** $u = \ln |z|$

16. $v = \ln |z|$ **17.** $u = x^3 - 3xy^2$

18. $u = 1/(x^2 + y^2)$ **19.** $v = (x^2 - y^2)^2$

20. $u = \cos x \cosh y$ **21.** $u = e^{-x} \sin 2y$

22–24 Determine a, b, c such that the given functions are harmonic and find a harmonic conjugate.

22. $u = e^{3x} \cos ay$ **23.** $u = \sin x \cosh cy$

24. $u = ax^3 + by^3$

25. (Harmonic conjugate) Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

26. TEAM PROJECT. Conditions for $f(z) = const.$ Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = const.$

(a) $\text{Re}\, f(z) = const$

(b) $\text{Im}\, f(z) = const$

(c) $f'(z) = 0$

(d) $|f(z)| = const$ (see Example 3)

27. (Two further formulas for the derivative). Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$(11) \quad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

28. CAS PROJECT. Equipotential Lines. Write a program for graphing equipotential lines $u = const$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to (a) $u = x^2 - y^2$, $v = 2xy$, (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$, (c) $u = e^x \cos y$, $v = e^x \sin y$.

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

$$(1) \qquad e^z = e^x(\cos y + i \sin y).$$