11. WRITING PROJECT. Sets in the Complex Plane. Extend the part of the text on sets in the complex plane by formulating that part in your own words and including examples of your own and comparing with calculus when applicable.

COMPLEX FUNCTIONS AND DERIVATIVES

12–15 **Function Values.** Find Re f and Im f. Also find their values at the given point z.

12. $f = 3z^2 - 6z + 3i,\ z = 2 + i$

13. $f = z/(z + 1),\ z = 4 - 5i$

14. $f = 1/(1 - z),\ z = \frac{1}{2} + \frac{1}{4}i$

15. $f = 1/z^2,\ z = 1 + i$

16–19 **Continuity.** Find out (and give reason) whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

16. $[\text{Re}\,(z^2)]/|z|^2$

17. $[\text{Im}\,(z^2)]/|z|$

18. $|z|^2\ \text{Re}\,(1/z)$

19. $(\text{Im}\,z)/(1 - |z|)$

20–24 **Derivative.** Differentiate

20. $(z^2 - 9)/(z^2 + 1)$

21. $(z^3 + i)^2$

22. $(3z + 4i)/(1.5iz - 2)$

23. $i/(1 - z)^2$

24. $z^2/(z + i)^2$

25. CAS PROJECT. Graphing Functions. Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves Re $f(z) = const$ and Im $f(z) = const$ in the same figure, and the curves $|f(z)| = const$ in another figure, where (a) $f(z) = \ldots$, (b) $f(z) = 1/z$, (c) $f(z) = z^4$.

26. TEAM PROJECT. Limit, Continuity, Derivative. **(a) Limit.** Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re}\, f(z) = \text{Re}\, l, \qquad \lim_{z \to z_0} \text{Im}\, f(z) = \text{Im}\, l.$$

(b) Limit. If $\lim_{z \to z_0} f(z)$ exists, show that this limit is unique.

(c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

(d) Continuity. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

(e) Differentiability. Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any z. Can you find other such functions?

(f) Differentiability. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

13.4 Cauchy–Riemann Equations. Laplace's Equation

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$\textbf{(1)} \qquad u_x = v_y, \qquad u_y = -v_x$$

[4]The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GR9] in App. 1.