EXAMPLE 6 A Geometric Application

Geometric problems may also lead to initial value problems. For instance, find the curve through the point (1, 1) in the xy-plane having at each of its points the slope $-y/x$.

Solution. The slope y' should equal $-y/x$. This gives the ODE $y' = -y/x$. Its general solution is $y = c/x$ (see Example 1). This is a family of hyperbolas with the coordinate axes as asymptotes.

Now, for the curve to pass through (1, 1), we must have $y = 1$ when $x = 1$. Hence the initial condition is $y(1) = 1$. From this condition and $y = c/x$ we get $y(1) = c/1 = 1$; that is, $c = 1$. This gives the particular solution $y = 1/x$ (drawn somewhat thicker in Fig. 5). ■

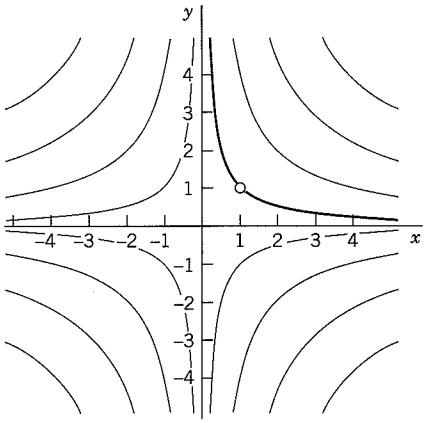

Fig. 5. Solutions of $y' = -y/x$ (hyperbolas)

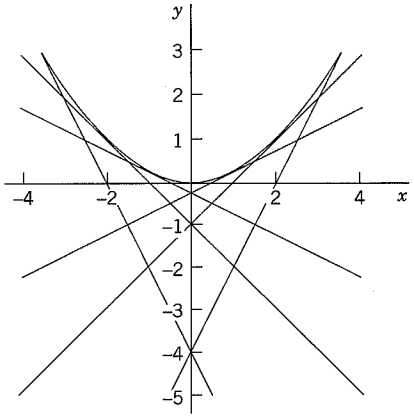

Fig. 6. Particular solutions and singular solution in Problem 16

PROBLEM SET 1.1

1–4 CALCULUS

Solve the ODE by integration.

1. $y' = -\sin \pi x$

2. $y' = e^{-3x}$

3. $y' = xe^{x^2/2}$

4. $y' = \cosh 4x$

5–9 VERIFICATION OF SOLUTION

State the order of the ODE. Verify that the given function is a solution. (a, b, c are arbitrary constants.)

5. $y' = 1 + y^2$, $y = \tan (x + c)$

6. $y'' + \pi^2 y = 0$, $y = a \cos \pi x + b \sin \pi x$

7. $y'' + 2y' + 10y = 0$, $y = 4e^{-x} \sin 3x$

8. $y' + 2y = 4(x + 1)^2$, $y = 5e^{-2x} + 2x^2 + 2x + 1$

9. $y''' = \cos x$, $y = -\sin x + ax^2 + bx + c$

10–14 INITIAL VALUE PROBLEMS

Verify that y is a solution of the ODE. Determine from y the particular solution satisfying the given initial condition. Sketch or graph this solution.

10. $y' = 0.5y$, $y = ce^{0.5x}$, $y(2) = 2$

11. $y' = 1 + 4y^2$, $y = \frac{1}{2} \tan (2x + c)$, $y(0) = 0$

12. $y' = y - x$, $y = ce^x + x + 1$, $y(0) = 3$

13. $y' + 2xy = 0$, $y = ce^{-x^2}$, $y(1) = 1/e$

14. $y' = y \tan x$, $y = c \sec x$, $y(0) = \frac{1}{2}\pi$

15. (Existence) (A) Does the ODE $y'^2 = -1$ have a (real) solution?

(B) Does the ODE $|y'| + |y| = 0$ have a general solution?

16. (Singular solution) An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution.* The ODE $y'^2 - xy' + y = 0$ is of the kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain Fig. 6.

17–22 MODELING, APPLICATIONS

The following problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. (Falling body) If we drop a stone, we can assume air resistance ("drag") to be negligible. Experiments show that under that assumption the acceleration $y'' = d^2y/dt^2$ of this motion is constant (equal to the so-called acceleration of gravity $g = 9.80$ m/sec^2 $= 32$ ft/sec^2). State this as an ODE for $y(t)$, the distance fallen as a function of time t. Solve the ODE to get the familiar law of free fall, $y = gt^2/2$.