EXAMPLE 6 A Geometric Application

Geometric problems may also lead to initial value problems. For instance, find the curve through the point (1, 1) in the xy-plane having at each of its points the slope \(-y/x\).

Solution. The slope \(y'\) should equal \(-y/x\). This gives the ODE \(y' = -y/x\). Its general solution is \(y = cx\) (see Example 1). This is a family of hyperbolas with the coordinate axes as asymptotes.

Now, for the curve to pass through (1, 1), we must have \(y = 1\) when \(x = 1\). Hence the initial condition is \(y(1) = 1\). From this condition and \(y = cx\) we get \(y(1) = c/1 = 1\); that is, \(c = 1\). This gives the particular solution \(y = 1/x\) (drawn somewhat thicker in Fig. 5).

![Fig. 5. Solutions of \(y' = -y/x\) (hyperbolas)](image)

![Fig. 6. Particular solutions and singular solution in Problem 16](image)

PROBLEM SET 1

1-4 CALCULUS
Solve the ODE by integration.
1. \(y' = -\sin \pi x\) 2. \(y' = e^{-3x}\)
3. \(y' = xe^{x^2/2}\) 4. \(y' = \cosh 4x\)

5-9 VERIFICATION OF SOLUTION
State the order of the ODE. Verify that the given function is a solution. \((a, b, c\) are arbitrary constants.)
5. \(y' = 1 + y^2, \quad y = \tan (x + c)\)
6. \(y'' + \pi^2 y = 0, \quad y = a \cos \pi x + b \sin \pi x\)
7. \(y'' + 2y' + 10y = 0, \quad y = 4e^{-x} \sin 3x\)
8. \(y' + 2y = 4(x + 1)^2, \quad y = 5e^{-2x} + 2x^2 + 2x + 1\)
9. \(y''' = \cos x, \quad y = -\sin x + ax^2 + bx + c\)

10-14 INITIAL VALUE PROBLEMS
Verify that \(y\) is a solution of the ODE. Determine from \(y\) the particular solution satisfying the given initial condition. Sketch or graph this solution.
10. \(y' = 0.5y, \quad y = ce^{0.5x}, \quad y(2) = 2\)
11. \(y' = 1 + 4y^2, \quad y = \frac{1}{2} \tan (2x + c), \quad y(0) = 0\)
12. \(y' = y - x, \quad y = ce^x + x + 1, \quad y(0) = 3\)
13. \(y' + 2xy = 0, \quad y = ce^{-x^2}, \quad y(1) = 1/e\)
14. \(y' = y \tan x, \quad y = c \sec x, \quad y(0) = \frac{1}{2}\pi\)

15. (Existence) (A) Does the ODE \(y'' = -1\) have a (real) solution?
   (B) Does the ODE \(|y'| + |y| = 0\) have a general solution?

16. (Singular solution) An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a singular solution. The ODE \(y'' - xy' + y = 0\) is of the kind. Show by differentiation and substitution that it has the general solution \(y = cx - c^2\) and the singular solution \(y = x^2/4\). Explain Fig. 6.

17-22 MODELING, APPLICATIONS
The following problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. (Falling body) If we drop a stone, we can assume air resistance ("drag") to be negligible. Experiments show that under that assumption the acceleration \(y'' = d^2y/dt^2\) of this motion is constant (equal to the so-called acceleration of gravity \(g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2\)).
   State this as an ODE for \(y(t)\), the distance fallen as a function of time \(t\). Solve the ODE to get the familiar law of free fall, \(y = gt^2/2\).