2.8 Exercises

**Practice Problems**

1. Let \( A = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \)
   
   Given \( A \), find a vector in Null \( A \) and a vector in Col \( A \).
   
   Each answer:
   
   \( \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} \) is in Null \( A \)? Is \( \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \) in Col \( A \)?

2. Suppose an \( n \times n \) matrix \( A \) is invertible. What can you say about \( \text{Col } A \) and about \( \text{Null } A \)?

3. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \).
   
   Draw a picture whose span is not in \( H \), of a vector in \( H \) with a scalar multiple.

4. Let \( H \) be a subspace of \( \mathbb{R}^3 \). For instance, any two vectors in \( H \) bounding the line. In each case, give a specific reason why the set of exercise 4 displays itself in \( \mathbb{R}^2 \).

5. Assume the sets include the set of exercise 4.
9. With $A$ and $p$ as in Exercise 7, determine if $p$ is in Nul $A$.

10. With $u = (-2, 3, 1)$ and $A$ as in Exercise 8, determine if $u$ is in Nul $A$.

In Exercises 11 and 12, give integers $p$ and $q$ such that Nul $A$ is a subspace of $\mathbb{R}^p$ and Col $A$ is a subspace of $\mathbb{R}^q$.

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$

13. For $A$ as in Exercise 11, find a nonzero vector in Nul $A$ and a nonzero vector in Col $A$.

14. For $A$ as in Exercise 12, find a nonzero vector in Nul $A$ and a nonzero vector in Col $A$.

Determine which sets in Exercises 15–20 are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

15. \[
\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

16. \[
\begin{bmatrix} -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}
\]

17. \[
\begin{bmatrix} 0 \\ -5 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 3 \\ 4 \end{bmatrix}
\]

18. \[
\begin{bmatrix} 1 \\ -5 \\ 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -5 \end{bmatrix}
\]

19. \[
\begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \\ 5 \\ -8 \\ 2 \\ -7 \end{bmatrix}
\]

20. \[
\begin{bmatrix} 1 \\ -6 \\ 3 \\ 7 \\ -2 \\ 0 \\ 7 \\ 5 \\ 9 \end{bmatrix}
\]

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A subspace of $\mathbb{R}^n$ is any set $H$ such that (i) the zero vector is in $H$, (ii) $u$, $v$, and $u + v$ are in $H$, and (iii) $c$ is a scalar and $cu$ is in $H$.

b. If $v_1, \ldots, v_p$ are in $\mathbb{R}^n$, then Span $\{v_1, \ldots, v_p\}$ is the same as the column space of the matrix $[v_1 \cdots v_p]$.

c. The set of all solutions of a system of $m$ homogeneous equations in $n$ unknowns is a subspace of $\mathbb{R}^m$.

d. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

e. Row operations do not affect linear dependence relations among the columns of a matrix.

22. a. A subset $H$ of $\mathbb{R}^n$ is a subspace if the zero vector is in $H$.

b. Given vectors $v_1, \ldots, v_p$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbb{R}^n$.

c. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

d. The column space of a matrix $A$ is the set of solutions of $Ax = b$.

e. If $B$ is an echelon form of a matrix $A$, then the pivot columns of $B$ form a basis for Col $A$.

Exercises 23–26 display a matrix $A$ and an echelon form of $A$. Find a basis for Col $A$ and a basis for Nul $A$.

23. $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$, $\sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$, $\sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$, $\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$, $\sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

27. Construct a $3 \times 3$ matrix $A$ and a nonzero vector $b$ such that $b$ is in Col $A$, but $b$ is not the same as any one of the columns of $A$.

28. Construct a $3 \times 3$ matrix $A$ and a vector $b$ such that $b$ is not in Col $A$.

29. Construct a nonzero $3 \times 3$ matrix $A$ and a nonzero vector $b$ such that $b$ is in Nul $A$.

30. Suppose the columns of a matrix $A = [a_1 \cdots a_p]$ are linearly independent. Explain why $[a_1, \ldots, a_p]$ is a basis for Col $A$.

In Exercises 31–36, respond as comprehensively as possible, and justify your answer.
symbols, $Nul A = \{0\}$ and $Nul A = \{0\}$. The only solution is the zero vector. In symbols, $Col A = \{0\}$.

By definition, the columns of any matrix always span the column space. In this case, $Col A = \{0\}$.

If $A$ is invertible, then the columns of $A$ span $\mathbb{R}^n$ by the invertible matrix theorem. Therefore, $Col A = \mathbb{R}^n$. The only vector in both $Nul A$ and $Col A$ is the zero vector. For most $n \times n$ matrices, the zero vector is in both $Nul A$ and $Col A$. Hence, $\dim Nul A + \dim \text{rank } A = n$.

The result shows that $n$ is in $Nul A$. Deciding whether a solution exists requires more work. Reduce the augmented matrix $[A | n]$ to echelon form to determine whether there is a solution. 

$$[ \begin{array}{c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} ]$$

To determine whether $n$ is in $Nul A$, simply compute $AN$.

**Solution**

A basis for $Col A$ is:

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

**Questions**

31. Suppose $A$ is a $5 \times 5$ matrix whose column space is not full.

1. What can you say about the shape of an $m \times n$ matrix $A$ when
   - $m > n$?
   - $m = n$?
   - $m < n$?

2. When can you say that the null space of an $m \times n$ matrix $A$ is:
   - $\{0\}$?
   - $\mathbb{R}^n$?
   - $\mathbb{R}^{m-n}$?
   - $\mathbb{R}^\infty$?

3. If $A \in \mathbb{R}^{m \times n}$ and $null A = \{0\}$, what can you say about $A$?

4. If $A$ is a $6 \times 4$ matrix and $null A = \{0\}$, what can you say about $A$?

5. If $A$ is a $6 \times 4$ matrix whose column space is full, what can you say about $A$?
Numerical Notes

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of $x$ in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

In practical applications, the effective rank of a matrix $A$ is often determined from the singular value decomposition of $A$, to be discussed in Section 7.4.

Practice Problems

1. Determine the dimension of the subspace $H$ of $\mathbb{R}^3$ spanned by the vectors $v_1, v_2,$ and $v_3$. (First, find a basis for $H$.)

\[
\begin{align*}
  v_1 &= \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, & v_2 &= \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix}, & v_3 &= \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix}
\end{align*}
\]

2. Consider the basis $B = \left\{ \begin{bmatrix} 1 \\ .2 \\ 1 \end{bmatrix}, \begin{bmatrix} .2 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. If $[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, what is $x$?

3. Could $\mathbb{R}^3$ possibly contain a four-dimensional subspace? Explain.

2.9 Exercises

In Exercises 1 and 2, find the vector $x$ determined by the given coordinate vector $[x]_B$ and the given basis $B$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$, $[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. $B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $[x]_B = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

In Exercises 3–6, the vector $x$ is in a subspace $H$ with a basis $B = \{b_1, b_2\}$. Find the $B$-coordinate vector of $x$.

3. $b_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, $x = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

4. $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, $x = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

5. $b_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $b_2 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$

6. $b_1 = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, $x = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

7. Let $b_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} -7 \\ -2 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $B = \{b_1, b_2\}$. Use the figure to estimate $[w]_B$ and $[x]_B$. Confirm your estimate of $[x]_B$ by using it and $\{b_1, b_2\}$ to compute $x$. 

The rank command
of columns of \( A \).

d. The dimension of \( \text{Col } A \) is the number of pivot columns.

c. The dimension of \( \text{Col } A \) is the number of columns.

Exercises 9–12 display a matrix \( A \) and an equation form of \( A \).

\[ \begin{bmatrix}
0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
-7 & 0 & 3 & -1 \\
6 & 6 & 2 & -1
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-6 & 1 & 6 & 4 \\
-3 & 0 & 9 & 1 \\
4 & 6 & 2 & -1
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}
\]

\[ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
5 & 4 & 2 & 1 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 0 & 0 \\
7 & -7 & 0 & 0 \\
-2 & 6 & -3 & 0 \\
3 & 4 & 2 & 1
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\]
c. If a set of \( p \) vectors spans a \( p \)-dimensional subspace \( H \) of \( \mathbb{R}^n \), then these vectors form a basis for \( H \).

18. a. If \( B \) is a basis for a subspace \( H \), then each vector in \( H \) can be written in only one way as a linear combination of the vectors in \( B \).

b. If \( B = \{v_1, \ldots, v_p\} \) is a basis for a subspace \( H \) of \( \mathbb{R}^n \), then the correspondence \( x \mapsto [x]_B \) makes \( H \) look and act the same as \( \mathbb{R}^p \).

c. The dimension of \( \text{Null} \ A \) is the number of variables in the equation \( Ax = 0 \).

d. The dimension of the column space of \( A \) is rank \( A \).

e. If \( H \) is a \( p \)-dimensional subspace of \( \mathbb{R}^n \), then a linearly independent set of \( p \) vectors in \( H \) is a basis for \( H \).

In Exercises 19–24, justify each answer or construction.

19. If the subspace of all solutions of \( Ax = 0 \) has a basis consisting of three vectors and if \( A \) is a \( 5 \times 7 \) matrix, what is the rank of \( A \)?

20. What is the rank of a \( 4 \times 5 \) matrix whose null space is three-dimensional?

21. If the rank of a \( 7 \times 6 \) matrix \( A \) is 4, what is the dimension of the solution space of \( Ax = 0 \)?

22. Show that a set \( \{v_1, \ldots, v_s\} \) in \( \mathbb{R}^n \) is linearly independent if \( \text{dim} \text{Span} \{v_1, \ldots, v_s\} = s \).

23. If possible, construct a \( 3 \times 4 \) matrix \( A \) such that \( \text{null} \ A = 2 \) and \( \text{dim} \text{Col} \ A = 2 \).

24. Construct a \( 4 \times 3 \) matrix with rank 1.

25. Let \( A \) be an \( n \times p \) matrix whose column space is \( p \)-dimensional. Explain why the columns of \( A \) must be linearly independent.

26. Suppose columns 1, 3, 5, and 6 of a matrix \( A \) are linearly independent (but are not necessarily pivot columns) and the rank of \( A \) is 4. Explain why the four columns mentioned must be a basis for the column space of \( A \).

27. Suppose vectors \( b_1, \ldots, b_p \) span a subspace \( W \), and let \( \{a_1, \ldots, a_p\} \) be any set in \( W \) containing more than \( p \) vectors. Fill in the details of the following argument to show that \( \{a_1, \ldots, a_p\} \) must be linearly dependent. First, let \( B = [b_1 \ldots b_p] \) and \( A = [a_1 \ldots a_p] \).

a. Explain why for each vector \( a_j \), there exists a vector \( c_j \) in \( \mathbb{R}^p \) such that \( a_j = Bc_j \).

b. Let \( C = [c_1 \ldots c_p] \). Explain why there is a nonzero vector \( u \) such that \( Cu = 0 \).

c. Use \( B \) and \( C \) to show that \( Au = 0 \). This shows that the columns of \( A \) are linearly dependent.

28. Use Exercise 27 to show that if \( A \) and \( B \) are bases for a subspace \( W \) of \( \mathbb{R}^p \), then \( A \) cannot contain more vectors than \( B \), and, conversely, \( B \) cannot contain more vectors than \( A \).

29. [M] Let \( H = \text{Span} \{v_1, v_2\} \) and \( B = \{v_1, v_2\} \). Show that \( x \) is in \( H \), and find the \( B \)-coordinate vector of \( x \), where

\[
\begin{bmatrix}
11 \\
-5 \\
10 \\
7
\end{bmatrix}
= \begin{bmatrix}
14 \\
-8 \\
13 \\
10
\end{bmatrix}, \quad x = \begin{bmatrix}
19 \\
-13 \\
18 \\
15
\end{bmatrix}
\]

30. [M] Let \( H = \text{Span} \{v_1, v_2, v_3\} \) and \( B = \{v_1, v_2, v_3\} \). Show that \( B \) is a basis for \( H \) and \( x \) is in \( H \), and find the \( B \)-coordinate vector of \( x \), when

\[
\begin{bmatrix}
-6 \\
4 \\
-9 \\
4
\end{bmatrix}
= \begin{bmatrix}
-8 \\
-3 \\
-3 \\
-7
\end{bmatrix}, \quad v_3 = \begin{bmatrix}
3 \\
-8 \\
-3
\end{bmatrix}, \quad x = \begin{bmatrix}
4 \\
5 \\
7 \\
3
\end{bmatrix}
\]

Solutions to Practice Problems

1. Construct \( A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \) so that the subspace spanned by \( v_1, v_2, v_3 \) is the column space of \( A \). A basis for this space is provided by the pivot columns of \( A \).

\[
A = \begin{bmatrix}
2 & 3 & -1 \\
-8 & -7 & 6 \\
6 & -1 & -7
\end{bmatrix} \sim \begin{bmatrix}
2 & 3 & -1 \\
0 & 5 & 2 \\
0 & -10 & -4
\end{bmatrix} \sim \begin{bmatrix}
2 & 3 & -1 \\
0 & 5 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

The first two columns of \( A \) are pivot columns and form a basis for \( H \). Thus \( \dim H = 2 \).
The strategy in Example 3 of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros! So the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

**Numerical Note**

By today’s standards, a $25 \times 25$ matrix is small. Yet it would be impossible to calculate a $25 \times 25$ determinant by cofactor expansion. In general, a cofactor expansion requires over $n!$ multiplications, and $25!$ is approximately $1.5 \times 10^{25}$.

If a computer performs one trillion multiplications per second, it would have to run for over 500,000 years to compute a $25 \times 25$ determinant by this method. Fortunately, there are faster methods, as we’ll soon discover.

Exercises 19–38 explore important properties of determinants, mostly for the $2 \times 2$ case. The results from Exercises 33–36 will be used in the next section to derive the analogous properties for $n \times n$ matrices.

**Practice Problem**

Compute $| 5 -7 2 2 |$

$| 0 3 0 -4 |$

$| -5 -8 0 3 |$

$| 0 5 0 -6 |$

\[ 1, 7, 9, 7, 2, 9, 2, 2, 3, 2, 7, 7, 2, 7 \]

### 3.1 Exercises

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $| 3 0 4 |$

2. $| 0 5 1 |$

3. $| 2 -4 3 |$

4. $| 3 1 2 |$

5. $| 1 4 -1 |$

6. $| 2 3 -4 |$

7. $| 1 4 0 |$

8. $| 4 0 5 |$

9. $| 6 0 0 5 |$

10. $| 0 0 3 0 |$

11. $| 0 0 1 5 |$

12. $| 0 0 0 2 |$

13. $| 0 0 2 0 |$

Compute the determinants in Exercises 9–14 by cofactor expansion. At each step, choose a row or column that involves the least amount of computation.

9. $| 1 -2 5 2 |$

10. $| 2 -6 -7 5 |$

11. $| 3 5 -8 4 |$

12. $| 5 0 4 4 |$

13. $| 1 -2 3 -5 |$

14. $| 0 0 2 0 |$

15. $| 0 0 9 -1 |$

16. $| 0 0 5 2 |$

17. $| 0 0 3 0 |$
40. The cofactor expansion of a minor (also a minor) is the determinant of the matrix obtained by deleting the ith row and jth column of the (n × n) matrix A. The (i, j)-cofactor of A is the determinant of the (n − 1) × (n − 1) submatrix.

41. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Find the determinant of \( A \).

39. A minor of order 2 × 2 is called a 2 × 2 determinant. When finding each determinant, use the same method as above.

In Exercises 29 and 30, find each determinant.

38. Let \( A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \). Find the determinant of \( A \).

The value of the determinant of a matrix is determined by the signs of the entries in the sum of the products of the entries in each row (or column) and their corresponding minors.

42. Write a reason for your answer.

Use Exercises 25–28 to answer the questions in Exercises 31 and 32.

Exercises 25–30 (See Section 2.7)

Calculate the determinants of the element matrices given in Exercises 25–30.