On the numerical simulation of the constrained wave motion: a penalty approach

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1 Introduction

The main goal of this paper is to address the solution of a constrained wave system which has application in theoretical and applied physics. The numerical methodology relies on: the penalty treatment of the constraints and an energy preserving time discretization leading to a scheme which is essentially unconditionally stable and second order accurate, both being combined with a globally continuous piecewise linear finite element approximation. Numerical experiments confirm the properties of the computational methods discussed here.

2 Formulation of the problem

Let Ω be a bounded domain in R^2 with boundary Γ and let $\boldsymbol{u}(x,t) : R^2 \times (0,T) \to R^3$. We consider the following constrained wave problem

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \Delta \boldsymbol{u} + \lambda \boldsymbol{u} = 0, \quad |\boldsymbol{u}|^2 = 1 \text{ in } \Omega \times (0,T), \tag{1}$$

with Dirichlet boundary conditions

$$\boldsymbol{u}(x,t) = \boldsymbol{g}(x,t) \text{ on } \partial\Omega \times (0,T), \tag{2}$$

and initial conditions

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \frac{\partial \boldsymbol{u}}{\partial t}(x,0) = \boldsymbol{u}_1. \tag{3}$$

Further we suppose that $\boldsymbol{g} = \boldsymbol{g}(x)$ and

$$\begin{cases} \boldsymbol{u}_0(x) \cdot \boldsymbol{u}_1(x) = 0, \ |\boldsymbol{u}_0(x)|^2 = 1 & \text{for a.a. } x \in \Omega, \\ |\boldsymbol{g}(x)|^2 = 1 & \text{for a.a. } x \in \partial\Omega. \end{cases} \tag{4}$$

We approximate problem (1) by the wave equation with penalty term

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \Delta \boldsymbol{u} + \frac{1}{\varepsilon}(|\boldsymbol{u}|^2 - 1)\boldsymbol{u} = 0, \tag{5}$$

namely a kind of "Ginzburg-Landau wave equation". Using the methods from [4] we prove

Proposition 1 *For every fixed ε problem (2)-(5) has a unique solution $\boldsymbol{u} \in L_\infty(0,T;H^1(\Omega))^3$ with $\frac{\partial \boldsymbol{u}}{\partial t} \in L_\infty(0,T;L^2(\Omega))^3$, provided that $\boldsymbol{u}_0 \in H^1(\Omega)^3$, $\boldsymbol{u}_1 \in L^2(\Omega)^3$, $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^3$ and assumptions (4) are fulfilled.*

Let $E_\varepsilon(t) = \frac{1}{2}\int_\Omega |\frac{\partial \boldsymbol{u}(t)}{\partial t}|^2 dx + \frac{1}{2}\int_\Omega |\nabla \boldsymbol{u}(t)|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (|\boldsymbol{u}(t)|^2 - 1)^2 dx$ be the energy of the system. We show that model (2)-(5) is *energy preserving:*

$$\frac{d}{dt}E_\varepsilon(t) = 0 \ \ \forall t \geq 0. \tag{6}$$

Moreover, due to assumption (4) the energy $E_\varepsilon(0)$ doesn't depend upon the penalty parameter ε and reduces to $E(0) = \frac{1}{2}\int_\Omega |\boldsymbol{u}_1|^2 dx + \frac{1}{2}\int_\Omega |\nabla \boldsymbol{u}_0|^2 dx$. Thus, the energy preservation implies the following estimate for any $t > 0$

$$\int_\Omega (|\boldsymbol{u}|^2 - 1)^2 dx \leq 2\varepsilon\Big(\int_\Omega |\boldsymbol{u}_1|^2 dx + \int_\Omega |\nabla \boldsymbol{u}_0|^2 dx\Big). \tag{7}$$

3 Discretization of problem (5)

First we approximate the nonlinear wave problem (5) by a semidiscrete scheme

$$\begin{aligned}&\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^{n-1} - 2\boldsymbol{u}^n}{\Delta t^2} - \Delta(\frac{\boldsymbol{u}^{n+1} + 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{4})\\ &+\frac{1}{2\varepsilon}(|\frac{\boldsymbol{u}^n + \boldsymbol{u}^{n+1}}{2}|^2 + |\frac{\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2}|^2 - 2)(\frac{\boldsymbol{u}^{n+1} + 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{4}) = 0,\end{aligned} \tag{8}$$

with corresponding boundary conditions, scheme (8) being initialized with

$$\begin{cases}\boldsymbol{u}^0 = \boldsymbol{u}_0, \ \ \boldsymbol{u}^1 - \boldsymbol{u}^{-1} = 2\Delta t \boldsymbol{u}_1,\\ \boldsymbol{u}^1 - 2\boldsymbol{u}^0 + \boldsymbol{u}^{-1} = \Delta t^2 \Delta \boldsymbol{u}^0.\end{cases} \tag{9}$$

Proposition 2 *1) If the initial conditions $\boldsymbol{u}_1 \in L_p(\Omega)^3$ and $\Delta \boldsymbol{u}_0 \in L_p(\Omega)^3$ (for any $p > 2$), then problem (8) for any n has a solution such that $\frac{\boldsymbol{u}^{n+1} + 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{4} \in H^1(\Omega)^3$ and the following energy preserving property is valid:*

$$E_{n+1/2} = E_{n-1/2} \quad \text{for all } n \geq 1, \tag{10}$$

where the discrete energy $E_{n+1/2}$ is defined by

$$E_{n+1/2} = \frac{1}{2}\int_\Omega |\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t}|^2 dx + \frac{1}{2}\int_\Omega |\nabla \frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2}|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (|\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2}|^2 - 1)^2 dx.$$

2) If $\boldsymbol{u}^1 \in H^1(\Omega)^3$ and $\Delta \boldsymbol{u}_0 \in H^1(\Omega)^3$, then $\boldsymbol{u}^n \in H^1(\Omega)^3$ and the following estimate holds:

$$\int_\Omega \big(|\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2}|^2 - 1\big)^2 dx \leq c_1\varepsilon + c_2\Delta t^4 \tag{11}$$

with constants c_1, c_2 independent on Δt and ε. Moreover, there exists a constant c_0, independent on Δt and ε, such that for $\Delta t \leq c_0 \varepsilon$ problem (8) has a unique solution.

Of course the above energy relations show also the *unconditional stability* of scheme (8).

We construct a finite dimensional approximation of semidiscrete scheme (8) via finite difference or finite element approximation with the simplest quadrature formulas to avoid non-diagonal mass matrices. We obtain a fully discrete approximation $(8)_h$, which is similar to (8). Later we drop the subscript h from u_h^n. The fully discrete scheme inherits the main properties of the semi-discrete scheme, namely, unique solvability, energy preservation and analogue of estimate (11) (the integrals over Ω are replaced by corresponding weighted sums).

4 Solution method

Let $\boldsymbol{z} = (\boldsymbol{u}^{n+1} + 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1})/4$, then problem $(8)_h$ can be written as follows

$$(4\boldsymbol{z} - 4\boldsymbol{u}^n) - \Delta t^2 \Delta_h \boldsymbol{z} + \frac{\Delta t^2}{2\varepsilon}(|2\boldsymbol{z} - \boldsymbol{u}^{n-1/2}|^2 + |\boldsymbol{u}^{n-1/2}|^2 - 2)\boldsymbol{z} = 0, \qquad (12)$$

where $\boldsymbol{u}^{n-1/2} = (\boldsymbol{u}^n + \boldsymbol{u}^{n-1})/2$, and where $-\Delta_h$ is the discrete Laplace operator,

$$\frac{\boldsymbol{u}^1 + \boldsymbol{u}^{-1} - 2\boldsymbol{u}_0}{\Delta t^2} - \Delta_h \boldsymbol{u}_0 = 0, \quad \frac{\boldsymbol{u}^1 - \boldsymbol{u}^{-1}}{2\Delta t} = u_1. \qquad (13)$$

Equations (12)-(13) are completed by Dirichlet boundary conditions. From (13) it is easy to compute $\boldsymbol{u}^1$. Denoting the left-hand side of (12) by $F(\boldsymbol{z})$ one can show that $F(\boldsymbol{z})$ is differentiable and the matrix $F'(\boldsymbol{z})$ has the form

$$F'(\boldsymbol{z}) = (4 + \frac{\Delta t^2}{2\varepsilon}(|2\boldsymbol{z} - \boldsymbol{u}^{n-1/2}|^2 + |\boldsymbol{u}^{n-1/2}|^2 - 2))I - \Delta t^2 \Delta_h$$
$$+ \frac{4\Delta t^2}{\varepsilon}\boldsymbol{z}\boldsymbol{z}^T - \frac{2}{\varepsilon}\Delta t^2 \boldsymbol{z}(\boldsymbol{u}^{n-1/2})^T \equiv F_0'(\boldsymbol{z}) + \frac{2}{\varepsilon}\Delta t^2 \boldsymbol{z}(\boldsymbol{z} - \boldsymbol{u}^{n-1/2})^T \qquad (14)$$

with I the unit matrix. Matrix $F_0'(\boldsymbol{z})$ is symmetric (unlike $F'(\boldsymbol{z})$) and positive definite for $\Delta t \leq 2\sqrt{\varepsilon}$. Clearly, $F_0'(\boldsymbol{z})$ differs from $F'(\boldsymbol{z})$ by the term which is expected to be "small" when starting from a "good" initial guess.

We use a modification of the Newton method

$$F_0'(\boldsymbol{z}^k)(\boldsymbol{z}^{k+1} - \boldsymbol{z}^k) + F(\boldsymbol{z}^k) = 0, \quad k = 0, 1, 2, ... \qquad (15)$$

To implement problem (15) we use a conjugate gradient algorithm.

5 Numerical results

We have performed numerical experiments for (2)-(5) in $\Omega = (0,1) \times (0,1)$ by using finite difference in space for (8) with uniform mesh step size h. We

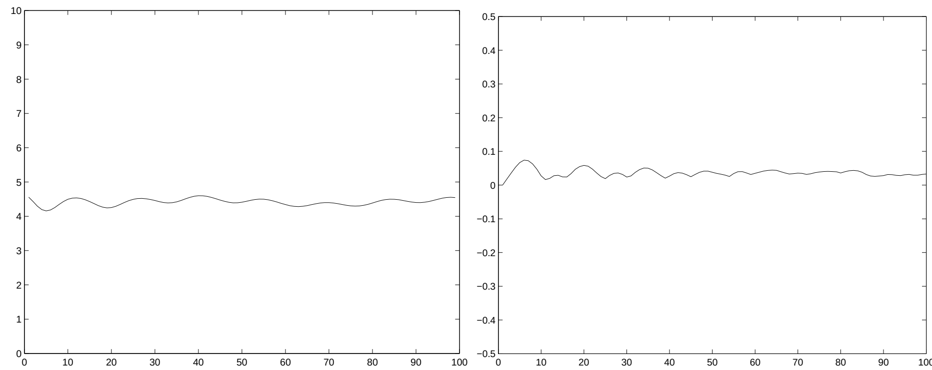

Fig. 1. Energy behavior (left) and $\max_x ||\boldsymbol{u}|^2 - 1|$ (right) for $\varepsilon = 10^{-3}$, $h = 1/50$, $\Delta t = 1/100$

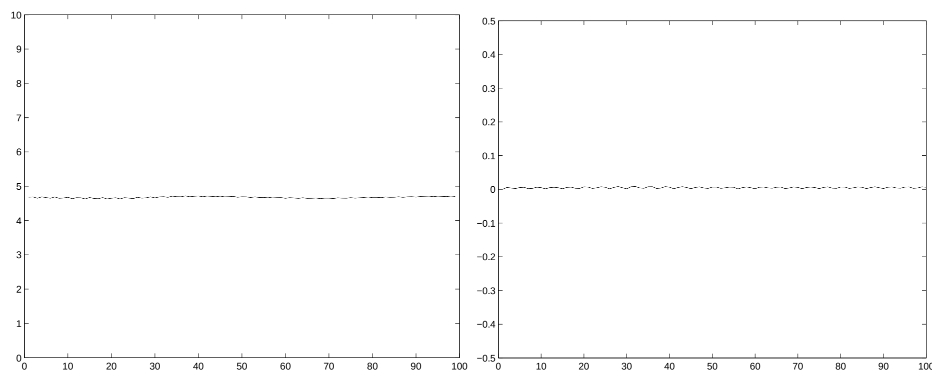

Fig. 2. Energy behavior (left) and $\max_x ||\boldsymbol{u}|^2 - 1|$ (right) for $\varepsilon = 10^{-5}$, $h = 1/50$, $\Delta t = 1/100$

considered different input data and different mesh and time step sizes h and Δt, and penalty parameter $\varepsilon > 0$. We have found that our numerical results agree very well with the constrained verification estimate (11). Moreover, our calculations showed that the discrete scheme is stable and that the iterative method (15) is convergent with a high rate of convergence under much less restrictive conditions between h, Δt and ε then those given by Propostion 2. The following graphs show the energy and $\max_x ||\boldsymbol{u}|^2 - 1|$ behavior in time, the shape of u_3 component of $\boldsymbol{u}$ for different values of ε with the initial data

$$u_0 = (\frac{x - x^*}{r}, \frac{y - y^*}{r}, \frac{\cos(\pi x - \pi y)}{r})^T, \quad \boldsymbol{u}_1 = 0,$$

$r = \sqrt{((x - x^*)^2 + (y - y^*)^2 + \cos^2(\pi x - \pi y))}$, $(x^*, y^*) = (1.1, 1.1)$ and boundary data consistent with initial one.

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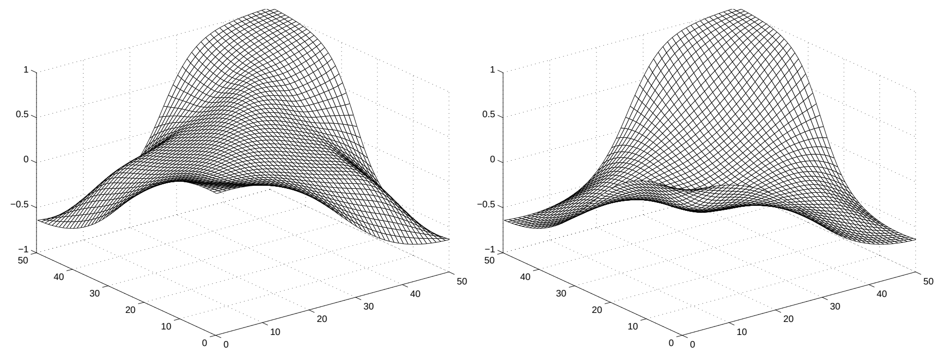

Fig. 3. u_3 component of $\boldsymbol{u}$ at $t = 1/4$ (left) and $t = 1$ (right) for $\varepsilon = 10^{-3}$, $h = 1/50$, $\Delta t = 1/100$

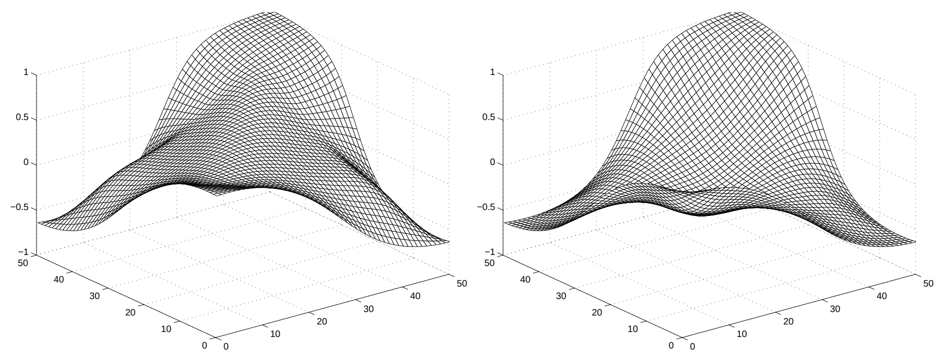

Fig. 4. u_3 component of $\boldsymbol{u}$ at $t = 1/4$ (left) and $t = 1$ (right) for $\varepsilon = 10^{-5}$, $h = 1/50$, $\Delta t = 1/100$

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