DETERMINATION OF $GL(3)$ CUSP FORMS BY CENTRAL VALUES OF $GL(3) \times GL(2)$ $L$-FUNCTIONS, LEVEL ASPECT

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Abstract. Let $f$ be a self-dual Hecke-Maass cusp form for $GL(3)$. We show that $f$ is uniquely determined by central values of $GL(2)$ twists of its $L$-function. More precisely, if $g$ is another self-dual $GL(3)$ Hecke-Maass cusp form such that $L \left( \frac{1}{2}, f \times h \right) = L \left( \frac{1}{2}, g \times h \right)$ for all $h \in S_{10}^*(q)$, for infinitely many primes $q$, then $f = g$.

1. Introduction

In their paper [LR], Luo and Ramakrishnan proved that a holomorphic cusp form is uniquely determined by the central values of twisted $L$-functions. More precisely, let $f(z) \in S_{2k}^{\text{new}}(N)$ and $g(z) \in S_{2k'}^{\text{new}}(N')$ be cuspidal normalized newforms of level $N$ and $N'$ respectively. They proved that if $L(1/2, f \times \chi_d) = cL(1/2, g \times \chi_d)$ for all quadratic characters $\chi_d$ with fundamental discriminant $d$ and some constant $c$, then $f = g$. Instead of twists by quadratic characters, Luo [Lu] considered twists by normalized newforms $h(z) \in S_{2l}^{\text{new}}(p)$ for all primes $p$ and proved the same result. Later Ganguly, Hoffstein and Sengupta [GHS] considered varying the twists $h(z)$ in the weight aspect and obtained the same result.

It is interesting to consider the analogous question for $GL(n)$ cusp forms. For a $GL(3)$ self-dual cusp form, such a result is true proved by Chinta and Diaconu [CD] with twists by quadratic characters and proved by the author [Liu] with twists by $GL(2)$ cusp forms in the weight aspect. In this paper we will prove the same result with twists by $GL(2)$ cusp forms in the level aspect (Theorem 1).

Let $S_{k}^{*}(q)$ denote the set of primitive cuspidal newforms on $\Gamma_0(q)$ with trivial nebentypus and weight $k$.

**Theorem 1.** Let $f$ and $g$ be self-dual Hecke-Maass forms for $SL(3, \mathbb{Z})$ of types $(\nu, \nu)$ and $(\mu, \mu)$ respectively. Let $c$ be a constant. If

$$L \left( \frac{1}{2}, f \times h \right) = cL \left( \frac{1}{2}, g \times h \right)$$

for all $h \in S_{10}^*(q)$, for infinitely many primes $q$, then $f = g$.

Key words and phrases. central values of $L$-functions, multiplicity one theorem.
As in [Liu], Theorem 1 follows from the asymptotic formulas for the first moments of twisted $L$-functions at the critical points (Theorem 3) and the multiplicity one theorem.

It is known that the symmetric square lift (Gelbart-Jacquet lift) of a $GL(2)$ Hecke cusp form $f$ is a self-dual $GL(3)$ cusp form. The following theorem is a consequence of our Theorem 1 and a result of Ramakrishnan [R].

**Theorem 2.** Let $f$ and $g$ be Hecke-Maass cusp forms for $SL(2, \mathbb{Z})$. Let $c$ be a constant. If
\[
\left(L\left(\frac{1}{2}, \text{sym}^2 f \times h\right)\right) = cL\left(\frac{1}{2}, \text{sym}^2 g \times h\right)
\]
for all $h \in S_{10}^\ast(q)$, for infinitely many primes $q$, then $f = g$.

2. Preliminaries

2.1. $GL(2)$ holomorphic cusp forms. For $k$ and $q$ integers, $k \geq 2$ and $\chi$ a character modulo $q$, let $S_k(q, \chi)$ denote the space of weight $k$ holomorphic cusp forms with level $q$ and nebentypus $\chi$. Let $H_k(q, \chi)$ be a normalized Hecke basis of $S_k(q, \chi)$. For each $h \in H_k(q, \chi)$, it has the Fourier expansion
\[
h(z) = \sum_{n=1}^{\infty} \lambda_h(n)n^{\frac{k-1}{2}}e(nz)
\]
where $e(z) = e^{2\pi i z}$ and $\lambda_h(1) = 1$. The following Petersson trace formula is well-known and can be found in [IK].

**Proposition 1** (Petersson trace formula).
\[
\sum_{h \in H_k(q, \chi)} \omega_h^{-1}\lambda_h(m)\overline{\lambda_h(n)} = \delta_{m,n} + 2\pi i^{k} \sum_{c \equiv 0 (\text{mod } q), c > 0} \frac{S_\chi(m, n; c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),
\]
where $\omega_h = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \|h\|^2$, $\delta_{m,n}$ equals 1 if $m = n$ and 0 otherwise, $S_\chi(m, n; c)$ is the Kloosterman sum with character $\chi$ defined below, and $J_{k-1}$ is the $J$-Bessel function.

The Kloosterman sum is defined as
\[
S_\chi(m, n; c) = \sum_{\tilde{a} \equiv 1 (\text{mod } c)} \chi(a)e\left(\frac{ma + n\tilde{a}}{c}\right),
\]
and for $\chi = 1$, A. Weil proved that
\[
|S(m, n; c)| \leq (m, n, c)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c),
\]
where $\tau(c)$ is the divisor function.
DETERMINATION OF \( GL(3) \) CUSP FORMS

Let \( S_k(q, \chi) \) denote the set of primitive newforms. In the case when \( \chi \) is primitive or \( k < 12 \) and \( \chi \) is trivial, we have \( H_k(q, \chi) = S_k^*(q, \chi) \).

For \( h \in S_k^*(q) \) with \( q \) squarefree and \( k < 12 \), the Hecke \( L \)-function of \( h \) is defined by

\[
L(s, h) = \sum_{n=1}^{\infty} \frac{\lambda_h(n)}{n^s} = \prod_p \left( 1 - \frac{\lambda_h(p)p^{-s} + \chi(p)p^{2s}}{-s} \right)^{-1}.
\]

It is entire and satisfies the functional equation

\[
\Lambda(s, h) = q^{s/2} \pi^{-s} \Gamma \left( \frac{s + k}{2} \right) \Gamma \left( \frac{s + k + 1}{2} \right) L(s, h) = \varepsilon_h \Lambda(1 - s, h)
\]

where \( \varepsilon_h = i^k \mu(q) \lambda_h(q) q^{1/2} = \pm 1 \) and \( \mu \) is the Möbius function.

**Proposition 2.** Let \( q \) be a prime. For any \( m, n \geq 1 \), we have

\[
2 \sum_{\substack{h \in S_{10}^*(q) \\
\varepsilon_h = 1}} \omega_h^{-1} \lambda_h(m) \lambda_h(n) = \delta_{m,n} - 2\pi \sum_{c=1}^{\infty} \frac{S(m, n; cq)}{cq} J_0 \left( \frac{4\pi \sqrt{mn}}{cq} \right)
\]

\[
+ q^{1/2} \left( \delta_{m,q} - 2\pi \sum_{c=1}^{\infty} \frac{S(m, q; cq)}{cq} J_0 \left( \frac{4\pi \sqrt{mnq}}{cq} \right) \right).
\]

**Proof.** Since \( \varepsilon_h = q^{1/2} \lambda_h(q) \), we have

\[
2 \sum_{\substack{h \in S_{10}^*(q) \\
\varepsilon_h = 1}} \omega_h^{-1} \lambda_h(m) \lambda_h(n) = \sum_{h \in S_{10}^*(q)} (1 + \varepsilon_h) \omega_h^{-1} \lambda_h(m) \lambda_h(n)
\]

\[
= \sum_{h \in S_{10}^*(q)} \omega_h^{-1} \lambda_h(m) \lambda_h(n) + q^{1/2} \sum_{h \in S_{10}^*(q)} \omega_h^{-1} \lambda_h(m) \lambda_h(q) \lambda_h(n).
\]

Because \( h \) is primitive and \( q \) is the level, \( \lambda_h(q) \lambda_h(n) = \lambda_h(nq) \) for all \( n \). Hence the result follows from the Petersson trace formula.

### 2.2. Automorphic forms on \( GL(3) \)

Here we use a classical setting for \( GL(3) \) Maass forms. One can see the details and notation in Goldfeld’s book [Go]. Let \( f \) be a normalized self-dual Hecke-Maass form for \( SL(3, \mathbb{Z}) \) of type \((\nu, \nu)\) and let \( A(m, n) \) be its \((m, n)\)-th Fourier coefficient. Note that \( A(1,1) = 1 \) and \( A(n,m) = A(m,n) \). Moreover we have (see [Go, Remark 12.1.8])

\[
(2.1) \quad \sum_{m^2n \leq N} |A(m,n)|^2 \ll_f N.
\]
By Cauchy’s inequality and (2.1), one can deduce that

\begin{equation}
\sum_{n \leq N} |A(m, n)| \ll f N|m|.
\end{equation}

The Godement-Jacquet $L$-function associated to $f$ is defined by

\begin{equation}
L(s, f) = \sum_{n=1}^{\infty} \frac{A(1, n)}{n^s} = \prod_{p} \left(1 - A(1, p)p^{-s} + A(p, 1)p^{-2s} - p^{-3s}\right)^{-1}.
\end{equation}

It is entire and satisfies the functional equation

\[ \gamma_{\nu}(s)L(s, f) = \gamma_{\nu}(1-s)L(1-s, f) \]

where

\[ \gamma_{\nu}(s) = \pi^{-3s} \frac{\Gamma\left(\frac{s+1-3\nu}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-1+3\nu}{2}\right)}{\Gamma\left(\frac{-s+1-\alpha}{2}\right)\Gamma\left(\frac{-s+1-\beta}{2}\right)\Gamma\left(\frac{-s+1-\gamma}{2}\right)}. \]

Set

\[ \alpha = -3\nu + 1, \ \beta = 0, \ \gamma = 3\nu - 1. \]

Suppose $k = 0, 1$ and $\psi$ is a smooth compactly supported function on $(0, \infty)$. Define

\[ \tilde{\psi}(s) := \int_{0}^{\infty} \psi(x)x^{s} \frac{dx}{x}. \]

For $\sigma > \max\{-1 - \Re\alpha, -1 - \Re\beta, -1 - \Re\gamma\}$, define

\[ \psi_{k}(x) := \frac{1}{2\pi i} \int_{\Re s = \sigma} (\pi^{3} x)^{-s} \frac{\Gamma\left(\frac{1+s+k+\alpha}{2}\right)\Gamma\left(\frac{1+s+k+\beta}{2}\right)\Gamma\left(\frac{1+s+k+\gamma}{2}\right)}{\Gamma\left(-\frac{s+k-\alpha}{2}\right)\Gamma\left(-\frac{s+k-\beta}{2}\right)\Gamma\left(-\frac{s+k-\gamma}{2}\right)} \tilde{\psi}(-s) ds, \]

\begin{equation}
\Psi_{\pm}(x) = \frac{1}{2\pi^{3/2}} \left( \psi_{0}(x) + \frac{1}{i} \psi_{1}(x) \right)
\end{equation}

and

\begin{equation}
\Psi_{-}(x) = \frac{1}{2\pi^{3/2}} \left( \psi_{0}(x) - \frac{1}{i} \psi_{1}(x) \right).
\end{equation}

The following Voronoi formula on $GL(3)$ was first proved by Miller and Schmid [MS] and a simpler proof was given by Goldfeld and Li [GL].
Proposition 3 ([MS], [GL]). Let \( \psi(x) \in C_c^\infty(0, \infty) \). Let \( d, d, c \in \mathbb{Z} \) with \( c \neq 0 \), \((c, d) = 1\), and \( d \equiv 1 \pmod{c}\). Then we have

\[
\sum_{n \geq 1} A(m, n) e\left(\frac{nd}{c}\right) \psi(n) = c \sum_{n_1 \mid cm \ n_2 > 0} A(n_2, n_1) S(md, n_2; mcn_1^{-1}) \Psi_+\left(\frac{n_2n_1^2}{c^3m}\right) + c \sum_{n_1 \mid cm \ n_2 > 0} A(n_2, n_1) S(md, -n_2; mcn_1^{-1}) \Psi_-\left(\frac{n_2n_1^2}{c^3m}\right).
\]

3. The twisted first moment of \( GL(3) \times GL(2) \) \( L \)-functions

Let \( h \in S_{10}^\ast(q) \) with \( q \) prime and let \( f \) be a self-dual Hecke-Maass form of type \((\nu, \nu)\). The Rankin-Selberg \( L \)-function of \( f \) and \( h \) is defined by

\[
L(s, f \times h) = \sum_{m \geq 1} \sum_{n \geq 1} \lambda_h(n) A(m, n) \frac{\lambda_h(n) A(m, n)}{(m^2n)^s}.
\]

It is entire and satisfies the functional equation

\[
\Lambda(s, f \times h) = \varepsilon_{f \times h} \Lambda(1 - s, f \times h)
\]

where \( \varepsilon_{f \times h} = \varepsilon_h = \pm 1 \) and

\[
\Lambda(s, f \times h) = q^{\frac{3}{2}s} \gamma(s) L(s, f \times h),
\]

\[
\gamma(s) = \pi^{-3s} \Gamma\left(\frac{s + \frac{9}{2} - \alpha}{2}\right) \Gamma\left(\frac{s + \frac{9}{2} - \beta}{2}\right) \Gamma\left(\frac{s + \frac{9}{2} - \gamma}{2}\right)
\]

\[
\times \Gamma\left(\frac{s + \frac{11}{2} - \alpha}{2}\right) \Gamma\left(\frac{s + \frac{11}{2} - \beta}{2}\right) \Gamma\left(\frac{s + \frac{11}{2} - \gamma}{2}\right).
\]

Remarks.
1. Luo-Rudnick-Sarnak [LRS] proved that \( |\Re\alpha|, |\Re\beta|, |\Re\gamma| \leq \frac{1}{2} - \frac{1}{10} \).
2. The above functional equation can be obtained by using the template in [Go, p.315].

Theorem 3. Let \( f \) be a self-dual Hecke-Maass form for \( SL(3, \mathbb{Z}) \), normalized such that \( A(1, 1) = 1 \). Then for any large prime \( q \), we have

\[
\sum_{h \in S_{10}^\ast(q) \atop \varepsilon_h = 1} \omega_h^{-1} L\left(\frac{1}{2}, f \times h\right) = L(1, f) + O(q^{-\frac{1}{8} + \varepsilon})
\]
and for any fixed prime $p$, we have

$$\sum_{h \in S_{10}(q)} \omega_h^{-1} L \left( \frac{1}{2}, f \times h \right) \lambda_h(p) = \left( \frac{A(1, p)}{p^{1/2}} - \frac{1}{p^{3/2}} \right) L(1, f) + O(q^{-\frac{1}{8} + \varepsilon}).$$

The implied constants depend on $\varepsilon$, $f$ and $p$.

Note that $L(1, f) \neq 0$ (see [JS]). We will prove Theorem 3 in Section 4. The proof is based on a general framework which uses the approximate functional equation and the Petersson trace formula. The main term comes from the diagonal term and the others contribute to error terms. To estimate the off-diagonal terms, we will need the Voronoi formula on $GL(3)$.

As explained in [Li], $L(\frac{1}{2}, f \times h) \geq 0$ by Lapid’s theorem [La]. For $q$ large enough such that the twisted first moment in Theorem 3 is nonvanishing, we have the following corollary.

Corollary 4. Let $f$ be a fixed self-dual Hecke-Maass form for $SL(3, \mathbb{Z})$. For each prime $q$ large enough, there exists $h \in S_{10}^*(q)$ such that $L(\frac{1}{2}, f \times h) > 0$.

4. Proof of Theorem 3

4.1. Approximate functional equation. Let $G(u) = e^{u^2}$. We have the following approximate functional equation (see [IK, Theorem 5.3]),

$$L \left( \frac{1}{2}, f \times h \right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \lambda_h(n) A(m, n) \frac{m^2 n}{(m^2 n)^{1/2}} V \left( \frac{m^2 n}{q^{3/2}} \right),$$

where

$$V(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(\frac{1}{2} + u)}{\gamma(\frac{1}{2})} \frac{du}{u}.$$  

Moreover the derivatives of $V(y)$ satisfy (see [IK, Proposition 5.4])

$$y^a V^{(a)}(y) \ll (1 + y)^{-A}.$$  

Here the implied constant depends on $a$ and $A$.

We will only show the second asymptotic formula in Theorem 3 since the proof of the first one is similar. By the approximate functional equation (4.1),
Lemma 1.

Proposition 2, and since $\delta_{p,nq} = 0$ for all $n$ if $q > p$, we have

$$\sum_{h \in \mathcal{S}_{10}(q)} \omega_h^{-1} L \left( \frac{1}{2}, f \times h \right) \lambda_h(p)$$

$$= \sum_{h \in \mathcal{S}_{10}(q)} \omega_h^{-1} 2 \sum_{m \geq 1} \sum_{n \geq 1} \lambda_h(n) A(m,n) \frac{1}{(m^2 n)^{1/2}} V \left( \frac{m^2 n}{q^{3/2}} \right) \lambda_h(p)$$

$$= \sum_{m \geq 1} \frac{A(m,p)}{(m^2 p)^{1/2}} V \left( \frac{m^2 p}{q^{3/2}} \right)$$

$$- 2\pi \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{(m^2 n)^{1/2}} V \left( \frac{m^2 n}{q^{3/2}} \right) \sum_{c=1}^{\infty} S(p,n; cq) \frac{J_q \left( \frac{4\pi \sqrt{m}}{cq} \right)}{cq}$$

$$- 2\pi q^{1/2} \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{(m^2 n)^{1/2}} V \left( \frac{m^2 n}{q^{3/2}} \right) \sum_{c=1}^{\infty} S(p,nq; cq) \frac{J_q \left( \frac{4\pi \sqrt{mq}}{cq} \right)}{cq}$$

$$:= D - E_1 - E_2.$$

4.2. Diagonal terms.

Lemma 1.

$$D = \left( \frac{A(1,p)}{p^{1/2}} - \frac{1}{p^{3/2}} \right) L(1, f) + O(q^{-\frac{3}{4}}).$$

Proof. Using the relation

$$A(m,n) = \sum_{d(m,n)} \mu(d) A \left( \frac{m}{d}, 1 \right) A \left( 1, \frac{n}{d} \right),$$

one can get

$$D = \frac{A(1,p)}{p^{1/2}} \cdot \frac{1}{2\pi i} \int_{(3)} \sum_{m \geq 1} \frac{A(m,1)}{m} \left( \frac{m^2 p}{q^{3/2}} \right)^{-u} G(u) \frac{\gamma(u + \frac{1}{2})}{\gamma(\frac{1}{2})} du$$

$$- \frac{1}{p^{3/2}} \cdot \frac{1}{2\pi i} \int_{(3)} \sum_{m \geq 1} \frac{A(m,1)}{m} \left( \frac{m^2 p^3}{q^{3/2}} \right)^{-u} G(u) \frac{\gamma(u + \frac{1}{2})}{\gamma(\frac{1}{2})} du$$

$$= \frac{A(1,p)}{p^{1/2}} \cdot \frac{1}{2\pi i} \int_{(3)} L(1 + 2u, f) \left( \frac{p}{q^{3/2}} \right)^{-u} G(u) \frac{\gamma(u + \frac{1}{2})}{\gamma(\frac{1}{2})} du$$

$$- \frac{1}{p^{3/2}} \cdot \frac{1}{2\pi i} \int_{(3)} L(1 + 2u, f) \left( \frac{p^3}{q^{3/2}} \right)^{-u} G(u) \frac{\gamma(u + \frac{1}{2})}{\gamma(\frac{1}{2})} du.$$
We shift the contour to \((-\frac{1}{11}\)) and only pick up a simple pole at \(u = 0\) (see Remarks 1). The residue at \(u = 0\) is
\[
\left( \frac{A(1,p)}{p^{1/2}} - \frac{1}{p^{3/2}} \right) L(1,f)
\]
and the integral over the line \(\Re u = -\frac{1}{11}\) is \(\ll q^{-3/22}\).

4.3. Off-diagonal terms. We will prove the off-diagonal terms, \(E_1\) and \(E_2\), only contribute the error terms. By taking smooth dyadic subdivisions (see [IM]), it suffices to estimate
\[
E_1,N := \sum_{m \geq 1} \sum_{n \geq 1} A(m,n) w\left( \frac{m^2 n}{N} \right) V\left( \frac{m^2 n}{q^{3/2}} \right) \sum_{c=1}^{\infty} S(p,n;cq) J_9\left( \frac{4\pi \sqrt{pmn}}{cq} \right)
\]
and
\[
E_2,N := q^{1/2} \sum_{m \geq 1} \sum_{n \geq 1} A(m,n) w\left( \frac{m^2 n}{N} \right) V\left( \frac{m^2 n}{q^{3/2}} \right) \sum_{c=1}^{\infty} S(p,nq;cq) J_9\left( \frac{4\pi \sqrt{pmnq}}{cq} \right)
\]
where \(w(\xi)\) is essentially a fixed smooth function with support contained in \([1,2]\) and \(N \ll q^{3/2+\varepsilon}\).

Lemma 2.
\[
E_{1,N} \ll q^{-1/2+\varepsilon}.
\]

Proof. For \(N \leq m^2 n \leq 2N\), we have
\[
x = \frac{4\pi \sqrt{pmn}}{cq} = \frac{4\pi \sqrt{p(m^2 n)^{1/2}}}{cq m} \leq \frac{4\pi \sqrt{p(2N)^{1/2}}}{cq} \ll q^{-1/4+\varepsilon}.
\]
Hence \(x < 1\) for \(q\) sufficient large. Recall that
\[
(4.4) \quad J_{k-1}(x) \ll \min(1, x^{k-1}).
\]
Using Weil’s bound for Kloosterman sums, (4.4), (4.3) and (2.2), we have
\[
E_{1,N} \ll \sum_{m \geq 1} \sum_{n \geq 1} |A(m,n)| \left| w\left( \frac{m^2 n}{N} \right) \right| \sum_{c \geq 1} (cq)^{-1/2+\varepsilon} \left( \frac{n^{1/2}}{cq} \right)^9
\ll q^{-19/2+\varepsilon} \sum_{N \leq m^2 n \leq 2N} \frac{|A(m,n)|}{(m^2 n)^{1/2}} n^{9/2}
\ll q^{-19/2+\varepsilon} N^6 \ll q^{-1/2+\varepsilon}.
\]
Lemma 3.

\[ E_{2,N} \ll q^{-1/8+\epsilon}. \]

Proof. Let

\[
R_1 = q^{1/2} \sum_{m \geq 1, n \geq 1} \frac{A(m, n)}{(m^2 n)^{1/2}} w \left( \frac{m^2 n}{N} \right) V \left( \frac{m^2 n}{q^{3/2}} \right) \times \\
\sum_{c > \frac{8 \sqrt{pq}}{q^{13/34} m}} \frac{S(p, nq; cq)}{cq} J_9 \left( \frac{4 \pi \sqrt{pmq}}{cq} \right)
\]

and

\[
R_2 = q^{1/2} \sum_{m \geq 1, n \geq 1} \frac{A(m, n)}{(m^2 n)^{1/2}} w \left( \frac{m^2 n}{N} \right) V \left( \frac{m^2 n}{q^{3/2}} \right) \times \\
\sum_{c \leq \frac{8 \sqrt{pq}}{q^{13/34} m}} \frac{S(p, nq; cq)}{cq} J_9 \left( \frac{4 \pi \sqrt{pmq}}{cq} \right).
\]

Then \( E_{2,N} = R_1 + R_2 \). We will show \( R_1 \ll q^{-1/8+\epsilon} \) and \( R_2 \) is negligible in the next two sections.

4.4. Estimate of \( R_1 \). For \( c > \frac{8 \sqrt{pq}}{q^{13/34} m} \) and \( N \leq m^2 n \leq 2N \), we have

\[
x = \frac{4 \pi \sqrt{pmq}}{cq} = \frac{4 \pi \sqrt{p(m^2 n)^{1/2}}}{cq^{1/2} m} \ll q^{-2/17}.
\]

Hence \( x < 1 \) for \( q \) sufficient large. Using Weil’s bound for Kloosterman sums, (4.4), (4.3) and (2.2), we have

\[
R_1 \ll q^{1/2} \sum_{m \geq 1, n \geq 1} \frac{|A(m, n)|}{(m^2 n)^{1/2}} \left| w \left( \frac{m^2 n}{N} \right) \right| \sum_{c > \frac{8 \sqrt{pq}}{q^{13/34} m}} (cq)^{-1/2+\epsilon} \left( \frac{\sqrt{pq}}{cq} \right)^9
\]

\[
\ll q^{-9/2+\epsilon} \sum_{N \leq m^2 n \leq 2N} \frac{|A(m, n)|}{(m^2 n)^{1/2}} n^{9/2} \sum_{c > \frac{8 \sqrt{pq}}{q^{13/34} m}} c^{-19/2+\epsilon}
\]

\[
\ll q^{-9/2+\epsilon} N^{-1/2} \sum_{N \leq m^2 n \leq 2N} |A(m, n)| n^{9/2} \left( \frac{N^{1/2}}{q^{13/34} m} \right)^{-17/2+\epsilon}
\]

\[
\ll q^{-5/4+\epsilon} N^{3/4+\epsilon} \ll q^{-1/8+\epsilon}.
\]
4.5. **Estimate of** $R_2$. To estimate $R_2$, we will use the Voronoi formula on $GL(3)$. After opening the Kloosterman sums in $R_2$, we will need to estimate the following sum over $n$,

$$H := \sum_{n \geq 1} A(m, n) \frac{w}{N^{1/2}} V \frac{m^2 n}{q^{3/2}} J_0 \frac{4\pi \sqrt{pq}}{cq} e \left( \frac{n\tilde{a}}{c} \right).$$

Applying the $GL(3)$ Voronoi formula (Proposition 3) with $\psi(y) = y - \frac{1}{2} w \frac{m^2 y}{N} V \frac{m^2 y}{q^{3/2}} J_9 \frac{4\pi \sqrt{pq}}{cq}$, we have

$$H = \sum_{n \geq 1} A(m, n)e \left( \frac{n\tilde{a}}{c} \right) \psi(n)$$

$$= c \sum_{n_2 | cm} \sum_{n_2 > 0} A(n_2, n_1) S(ma, n_2, mcn_1^{-1}) \Psi_+ \left( \frac{n_2 n_1^2}{c^3 m} \right)$$

$$+ c \sum_{n_1 | cm} \sum_{n_2 > 0} A(n_2, n_1) S(ma, -n_2, mcn_1^{-1}) \Psi_- \left( \frac{n_2 n_1^2}{c^3 m} \right)$$

where $\Psi_+(x)$ and $\Psi_-(x)$ are defined in (2.3) and (2.4).

As explained in [Li], $\psi_1(x)$ has similar asymptotic behavior to $\psi_0(x)$, so we only consider $\psi_0(x)$.

Since $c \leq \frac{8\pi \sqrt{pN}}{q^{3/4} N}$,

$$\frac{n_2 n_1^2}{c^3 m} \cdot \frac{N}{m^2} \gg q^{39/34} N^{-1/2} \gg q^{27/68 - \varepsilon}.$$ 

Hence we can apply [Li, Lemma 2.1] for $x = \frac{n_2 n_1^2}{c^3 m}$ which gives

$$\psi_0(x) = 2\pi^4 x \int_0^{\infty} \psi(y) \sum_{j=1}^{K} c_j \cos(6\pi x y^{1/3}) \sin \left( \frac{6\pi x y^{1/3} y^{1/3}}{(\pi^3 x y)^{1/3}} \right) dy$$

$$+ O \left( (q^{27/68-\varepsilon})^{-K/2} \right)$$

(4.5)

where $c_j$ and $d_j$ are some constants. In particular $c_1 = 0$ and $d_1 = -\frac{2}{\sqrt{3} \pi}$. We only deal with the first term since the other terms can be analyzed in a similar way. So we have to estimate the integral

$$x \int_0^{\infty} \psi(y) \frac{\sin(6\pi x y^{1/3} y^{1/3})}{(\pi^3 x y)^{1/3}} dy = x^{2/3} \int_0^{\infty} r(y) \sin(a(y)) dy$$

where

$$r(y) = y^{-5/6} w \left( \frac{m^2 y}{N} \right) V \frac{m^2 y}{q^{3/2}} J_0 \frac{4\pi \sqrt{pqy}}{cq}$$
and
\[ a(y) = 6\pi x^{1/3} y^{1/3}. \]

Using (4.3) and \( x^l J_{k-1}^{(l)} \ll 1 \), we have \( r'(y) \ll \left( \frac{m^2}{N} \right)^{11/6} \). Moreover we have
\[ a'(y) \gg q^{13/34} N^{2/3} m^{5/3} \gg q^{13/34} N^{-1/6} \gg q^{9/68 - \varepsilon}. \]

By integration by parts, one shows that \( H \) is negligible and so is \( R_2 \).

5. PROOFS OF THEOREMS 1 AND 2.

The proofs of Theorems 1 and 2 are the same as in [Liu]. For completeness, we include them here.

Proof of Theorem 1. Let \( A(m,n) \) and \( B(m,n) \) be the \((m,n)\)-th Fourier coefficients of \( f \) and \( g \) respectively. Then the assumption \( L(1/2, f \times h) = cL(1/2, g \times h) \) and Theorem 3 imply
\[ L(1, f) = cL(1, g) \]
and
\[ \left( \frac{A(1,p)}{p^{1/2}} - \frac{1}{p^{3/2}} \right) L(1, f) = \left( \frac{B(1,p)}{p^{1/2}} - \frac{1}{p^{3/2}} \right) cL(1, g) \]
for all primes \( p \). Hence \( A(1,p) = B(1,p) \) for all primes \( p \). So \( f = g \) by the multiplicity one theorem [JS1].

Proof of Theorem 2. Recall that (see [MS1] for example)
\[ L(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \lambda_f^{(2)}(n)n^{-s} \]
where
\[ \lambda_f^{(2)}(n) = \sum_{m^2 = n} \lambda_f(m^2). \]

Hence \( \lambda_f^{(2)}(p) = \lambda_f(p^2) \). Let \( F \) and \( G \) be the Gelbart-Jacquet lifts of \( f \) and \( g \) respectively. Then \( F \) and \( G \) are \( GL(3) \) self-dual Maass cusp forms which satisfy \( L(s, F) = L(s, \text{sym}^2 f) \) and \( L(s, G) = L(s, \text{sym}^2 g) \). Moreover their Fourier coefficients
\[ A_F(1,n) = \lambda_f^{(2)}(n), \quad A_G(1,n) = \lambda_g^{(2)}(n). \]

As in the proof of Theorem 1, our assumption gives us \( A_F(1,p) = A_G(1,p) \) for all prime \( p \) and hence \( \lambda_f(p^2) = \lambda_g(p^2) \) for all prime \( p \). Using the Hecke relation \( \lambda_f(p^2)^2 = \lambda_g(p^2) + 1 \), we have \( \lambda_f(p^2)^2 = \lambda_g(p^2)^2 \). Then we use (Ramakrishnan[R], Corollary 4.1.3) to conclude \( f = g \).
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References


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