Convergence, and Hausdorff and $T_1$ Spaces

In a topological space $(X, \tau)$, we say a countable sequence of points $\{x_n\}$ converges (or $\tau$-converges) to a point $x \in X$ iff for each $\tau$-nbhd $U$ of $x$, there exists a corresponding positive integer $N$ such that $x_n \in U$ for all $n \geq N$. In familiar spaces such as $\mathbb{R}$ and $\mathbb{R}^2$ with the usual topology, sequences can converge to at most one point. However, this is not true in a general topological space. At this point, one might be asking, “For what types of topological spaces is it true that sequences can converge to only one point?” The answer lies in the following discussion of the so-called Hausdorff and $T_1$ properties.

A topological space $(X, \tau)$ is called a Hausdorff space (or is said to be Hausdorff, or to have the Hausdorff property) iff for every distinct pair of points $x_1$ and $x_2$ in $X$, there exists a nbhd $U_1$ of $x_1$ and a nbhd $U_2$ of $x_2$ such that $U_1 \cap U_2 = \emptyset$. On page 99, Munkres proves the following:

In a Hausdorff space $X$, every finite point set is closed.

The property that every finite point set is closed is not equivalent to being Hausdorff, however. For example (as Munkres points out), in the space $(\mathbb{R}, \tau_{cof})$, every finite point set is closed, but this space is not Hausdorff. This condition that all finite point sets be closed is called the $T_1$ axiom (or “condition”, or “property”). The relationship between these properties (Hausdorff and $T_1$) and convergence is addressed in the following theorems.

Munkres, Thm 17.9 Let $(X, \tau)$ be a $T_1$ topological space, and let $A \subseteq X$. Then the point $x$ is a limit point of $A$ iff every $\tau$-nbhd of $x$ contains infinitely many points of $A$.

Munkres, Thm 17.10 If $(X, \tau)$ is a Hausdorff space, then a sequence of points of $X$ $\tau$-converges to at most one point of $X$.
There is generally much more focus on Hausdorff spaces than on $T_1$ spaces, since every Hausdorff space is also $T_1$, and it is the property of convergence of sequences to at most one point that is shared by most spaces that are encountered in application.

**Closure Operators**

In the early days of topology (1900-1930), various definitions were proposed by various mathematicians to deal with such notions as limit, continuity, and convergence in a systematic way. One such alternative approach to topology is described in the following definition.

**Def N8.1** Let $X$ be a set and $\psi : 2^X \rightarrow 2^X$ be a "set function" satisfying the following conditions:

(C1) $\psi(\phi) = \phi$;

(C2) $A \subseteq \psi(A)$ for all $A \in 2^X$;

(C3) $\psi(A \cup B) = \psi(A) \cup \psi(B)$ for all $A, B \in 2^X$;

(C4) $\psi(\psi(A)) = \psi(A)$ for all $A \in 2^X$.

Then $\psi$ is called a **closure operator** on $X$, and $(X, \psi)$ is called a **closure space**.

E. Čech used the terms "closure operator" and "closure space" requiring only the first 3 of the above axioms. The axiom (C4), due to Kuratowski, is needed to establish the close relationship between "closure spaces" and "topological spaces" which shall next be described.

In any topological space $(X, \tau)$, recall the notation $\text{Cl}_\tau A$ for the smallest closed overset of $A$; note that $\text{Cl}_\tau : 2^X \rightarrow 2^X$.

**Prop N8.1** In a top. space $(X, \tau)$, $\text{Cl}_\tau$ is a closure operator on $X$, as defined above.
Proof:

(C1) \( \text{Cl}_r \phi = \phi \), since \( \phi \) is the smallest closed overset of \( \phi \).

(C2) \( A \subseteq \text{Cl}_r A \) follows immediately from the description of \( \text{Cl}_r A \) as the smallest closed overset of \( A \).

(C3) We need to prove \( \text{Cl}_r (A \cup B) = (\text{Cl}_r A) \cup (\text{Cl}_r B) \). Since closure preserves containment (i.e. \( G \subseteq H \implies \text{Cl}_r G \subseteq \text{Cl}_r H \)), we have \( (A \subseteq A \cup B \implies \text{Cl}_r A \subseteq \text{Cl}_r (A \cup B)) \) and \( (B \subseteq A \cup B \implies \text{Cl}_r B \subseteq \text{Cl}_r (A \cup B)) \). From these two inclusions, it follows that \( (\text{Cl}_r A) \cup (\text{Cl}_r B) \subseteq \text{Cl}_r (A \cup B) \). To prove the opposite inclusion, let \( x \in \text{Cl}_r (A \cup B) \). By Prop 3 from our previous set of notes, \( U \cap (A \cup B) \neq \emptyset \) for all nbhds \( U \) of \( x \). If \( x \not\in \text{Cl}_r A \), then there is a nbhd \( W \) of \( x \) such that \( W \cap A = \emptyset \). If \( x \not\in \text{Cl}_r B \), then there is a nbhd \( V \) of \( x \) such that \( V \cap A = \emptyset \). With \( U = W \cap V \), then \( U \) would be a nbhd of \( x \) such that \( U \cap (A \cup B) = \emptyset \), which is a contradiction of the fact that \( x \in \text{Cl}_r (A \cup B) \). So we must have either \( x \in \text{Cl}_r A \) or \( x \in \text{Cl}_r B \).

(C4) \( \text{Cl}_r A \subseteq \text{Cl}_r (\text{Cl}_r A) \) follows by (C2). For the reverse inclusion, let \( x \in \text{Cl}_r (\text{Cl}_r A) \). Then for any nbhd \( U \) of \( x \), \( U \cap \text{Cl}_r A \neq \emptyset \). This means there is a \( y \in U \cap \text{Cl}_r A \). But \( U \) is a nbhd of \( y \), and since \( y \in \text{Cl}_r A \), we must have \( U \cap A \neq \emptyset \). Since this holds for any nbhd of \( x \), we have \( x \in \text{Cl}_r A \), as desired.

Here is a converse to Prop N8.1 which we will state without taking the time to prove it.

Prop N8.2 Let \( (X, \psi) \) be a closure space, and let \( \gamma = \{ A \subseteq X \mid A = \psi(A) \} \). Then \( \gamma \) satisfies the closed set axioms for a topology (i.e. \( \tau = \{ X - A \mid A \in \psi \} \) is a topology on \( X \)). Furthermore, the closure operator \( \text{Cl}_{\tau} \) derived from \( \tau \) equals the original closure operator \( \psi \).
Propositions 1 and 2 together lead to the conclusion that there is a one-to-one correspondence between the set of all closure operators on \(X\) and the set \(\Pi(X)\) of all topologies on \(X\). Thus, for all practical purposes, "closure spaces" are the same as "topologies". The preceding results can be essentially duplicated using "interior operators" rather than "closure operators", though we shall not discuss this approach.

**Example 1** Let \(G\) be any group, with identity element \(e\). For each subset \(A \subseteq G\), let \(\psi(A)\) be the smallest subgroup of \(G\) containing \(A\). Which of the properties of a closure operator are satisfied by \(\psi\)?

**Example 2** Let \(\psi\) be the set function on \(\mathbb{R}\) defined as follows:

\[
\psi(\emptyset) = \emptyset;
\]

\[
\psi(A) = \mathbb{R} \text{ if } A \text{ has no lower bound};
\]

\[
\psi(A) = [a, \infty) \text{ if } a = \inf A.
\]

One may verify that \(\psi\) is a closure operator on \(\mathbb{R}\), relative to which the closed sets are \(\{\emptyset, \mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}\). The associated topology is \(\tau_o = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}\). This topology has some interesting properties which make it worth including in our "prefered" family of topologies on \(\mathbb{R}\). The lattice of all these topologies is shown on the right.

**Set Theoretic Facts About Functions**

Our next major topic will be continuous functions, but prior to that, it will be useful to state a theorem summarizing the behavior of functions relative to unions, intersections, and set complements.
Prop N8.3 Let \( f : X \to Y \), and let \( f^{-1} \subseteq Y \times X \) be the inverse relation. Let \( A \subseteq X \), \( B \subseteq Y \), \( \{A_i \mid i \in I\} \subseteq 2^X \), \( \{B_j \mid j \in J\} \subseteq 2^Y \). Then the following hold:

1. \( f(\cup A_i) = \cup f(A_i) \)
2. \( f(\cap A_i) \subseteq \cap f(A_i) \) (equality if \( f \) is injective)
3. \( f(X - A) \supseteq \cup f(X) - f(A) \)
4. \( f^{-1}(\cup B_j) = \cup f^{-1}(B_j) \)
5. \( f^{-1}(\cap B_j) \subseteq \cap f^{-1}(B_j) \)
6. \( f^{-1}(Y - B) = X - f^{-1}(B) \)
7. \( f(f^{-1}(B)) \subseteq B \) (equality if \( f \) is surjective)
8. \( A \subseteq f^{-1}(f(A)) \) (equality if \( f \) is injective).

Continuous Functions

Let \((X, \tau)\) and \((Y, \mu)\) be two topological spaces, and let \( f : X \to Y \) be a function (also called a "map" or a "mapping"). If \( a \in X \), we say that \( f \) is continuous at \( a \) iff for every \( \mu \)-nbhd \( V \) of \( f(a) \), there exists a \( \tau \)-nbhd \( U \) of \( a \) such that \( f(U) \subseteq V \). If \( A \subseteq X \) and \( f \) is continuous at every \( a \in A \), then we say \( f \) is continuous on \( A \). If \( f \) is continuous at every \( a \in X \), we simply say that \( f : (X, \tau) \to (Y, \mu) \) is continuous. Some equivalent characterizations of continuity are given in the next theorem.

Prop N8.4 Let \( f : (X, \tau) \to (Y, \mu) \). The following are equivalent:

1. \( f \) is continuous.
2. If \( V \) is \( \mu \)-open, then \( f^{-1}(V) \) is \( \tau \)-open.
3. If \( B \) is \( \mu \)-closed, then \( f^{-1}(B) \) is \( \tau \)-closed.
4. If \( A \subseteq X \), then \( f(\text{Cl}_\tau A) \subseteq \text{Cl}_\mu f(A) \).
Proof:

(1) $\implies$ (2): Suppose $f$ is continuous and $V$ is $\mu$-open, and let $x \in f^{-1}(V)$. Then $f(x) \in V$, so by continuity of $f$, $\exists U_x \in \tau$ such that $f(U_x) \subseteq V$. Thus we can write $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$, which shows that $f^{-1}(V) \in \tau$.

(2) $\implies$ (3): Assume (2), and suppose $B \subseteq Y$ is $\mu$-closed. Then $Y - B$ is $\mu$-open, and so by (2), $f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$ is $\tau$-open, which implies that $f^{-1}(B)$ is $\tau$-closed.

(3) $\implies$ (4): Assume (3), and let $A \subseteq X$. Then $\operatorname{Cl}_\mu f(A)$ is a $\mu$-closed set, so by (3), $f^{-1}(\operatorname{Cl}_\mu f(A))$ is a $\tau$-closed set which contains $A$, and therefore $f^{-1}(\operatorname{Cl}_\mu f(A))$ must contain $\operatorname{Cl}_\tau A$ as well. Thus we have

$$\operatorname{Cl}_\tau A \subseteq f^{-1}(\operatorname{Cl}_\mu f(A)) \implies f(\operatorname{Cl}_\tau A) \subseteq f(f^{-1}(\operatorname{Cl}_\mu f(A))) \subseteq \operatorname{Cl}_\mu f(A),$$

where the last $\subseteq$ is by Prop N8.2 (7).

(4) $\implies$ (1): Assume (4). To prove $f$ is continuous, let $a \in A$ and let $V$ be a nbhd of $f(a)$. Let $A = f^{-1}(Y - V)$. Then $a \not\in A$, and $f(a) \not\in \operatorname{Cl}_\mu f(A)$, since $V \cap f(A) = \phi$. By (4), since $f(a) \not\in \operatorname{Cl}_\mu f(A)$, we also have $f(a) \not\in f(\operatorname{Cl}_\tau A)$, which means that $a \not\in \operatorname{Cl}_\tau A$, and so $U = X - \operatorname{Cl}_\tau A$ is a $\tau$-nbhd of $a$. Finally, we show that $f(U) \subseteq V$. To do this, let $y \in f(U)$. Then $y = f(x)$ for some $x \in X - \operatorname{Cl}_\tau A$, and we have the following string of implications:

$$x \in X - \operatorname{Cl}_\tau A \implies x \in X - A \implies x \not\in f^{-1}(Y - V) \implies y \not\in Y - V \implies y \in V.$$

Therefore $f(U) \subseteq V$, and $f$ is continuous at $a$.

HW 4 (due Monday 9/19, anytime):

§16 (p. 91): 4, 10

§17 (p. 100): 4, 6, 8, 11, 15, 16

§18 (p. 111): 2, 5, 6, 7