Math 490 Notes 16

Sequential Convergence (continued)

In this set of notes, we conclude our discussion of sequential convergence. The following lemma will be useful in proving the next proposition.

Lemma N16.1 Let (X,τ) be a first countable topological space. For each $x \in X$, there is a countable, nested nbhd basis $\{U_n\}$ of x such that $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$. If $x_n \in U_n$ for all n, then $(x_n) \to x$.

Proof: For $x \in X$, let $\{V_n \mid n \in \mathbb{N}\}$ be a countable nbhd basis at x (which exists by the first countability assumption), and let $U_1 = V_1$, $U_2 = V_1 \cap V_2$, ..., $U_n = V_1 \cap V_2 \cap \ldots \cap V_n$, ... Then $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ is the desired nested nbhd basis at x. If $x_n \in U_n$ for all n, then since the U_n's are nested and form a nbhd basis, it's (hopefully) clear that (x_n) is eventually in any nbhd U of x, which means $(x_n) \to x$. ■

Prop N16.1 Let (X,τ) be a topological space.

(a) If (X,τ) is T_2, then sequential limits are unique.

(b) If (X,τ) is first countable, then (X,τ) is T_2 iff every convergent sequence has a unique limit.

Proof: Part of HW 7. Here's a hint for (b): If (X,τ) is not T_2, use the lemma to construct a sequence which converges to two different points.

Recall that if set X is uncountable, then in (X,τ_{coc}), the only convergent sequences are the almost constant sequences. This example shows that Prop N16.1(b) is false without the assumption of first countability. Recall also that if X is infinite, then in (X,τ_{cof}), any sequence which has no constant subsequence converges to every point in X. This example shows that convergent sequences in T_1 spaces need not have unique limits. One can, however, prove the following.

Prop N16.2 A topological space (X, τ) is T_1 iff every almost constant sequence converges only to the given constant.

Proof: Part of HW 7.

Prop N16.3 Let (X, τ) be a topological space, and let A,B be subsets of X.

(a) If (x_n) is a sequence in B and $(x_n) \to x$, then $x \in \operatorname{Cl}_\tau B$.

(b) If (X, τ) is first countable, then $\operatorname{Cl}_\tau B = \{x \in X \mid \exists (x_n) \to x \text{ with } x_n \in B \text{ for all } n\}$.

(c) If A is closed, $x_n \in A$ for all n, and $(x_n) \to x$, then $x \in A$.

(d) If (X, τ) is first countable, then A is closed iff $x \in A$ whenever $(x_n) \to x$ and $x_n \in A$ for all n.

Proof:

(a) Assume $(x_n) \to x$ with $x_n \in B$ for all n. Then for any nbhd U of x, (x_n) is eventually in U (by def. of convergence), and since $x_n \in B$ for all n, it follows that $U \cap B \neq \phi$. Since U was arbitrary, $x \in \operatorname{Cl}_\tau B$.

(b) Part of HW 7.

(c) Part of HW 7.

(d) ($\Rightarrow$) follows by (c), and I believe ($\Leftarrow$) follows from (b). ■

In any topological space, the closed sets uniquely determine the topology, and for first countable spaces, it follows from Prop N16.3(d) that the convergence of sequences uniquely determines the topology. This is not true in general, since (as we've seen) the discrete and cocountable topologies on an uncountable set have the same sequential convergence, but the topologies are quite different. Note that τ_{d} is first-countable while τ_{coc} is not. Prop N16.4(d) assures us that there is no other first countable topology with the same sequential convergence as τ_{d}.

Prop N16.4 Let $f : (X, \tau) \to (Y, \mu)$ be a function between topological spaces.

(a) If f is continuous and $(x_n) \xrightarrow{\tau} x$, then $(f(x_n)) \xrightarrow{\mu} f(x)$.

(b) If (X, τ) is first countable, then f is continuous iff $(x_n) \xrightarrow{\tau} x \Rightarrow (f(x_n)) \xrightarrow{\mu} f(x)$.

Proof: Essentially the same as that of Theorem 21.3 in Munkres. Munkres proves many of these ideas with first countability replaced by metrizability. Since metrizability implies first countability, our results imply his. He does remark about this at the top of page 131.

Prop N16.5 Let $(X, \tau) = \prod_{i \in I} (X_i, \tau_i)$. If $(\mathbf{x}_n)$ is a sequence in X, then $(\mathbf{x}_n) \xrightarrow{\tau} \mathbf{x}$ iff $(p_i(\mathbf{x}_n)) \xrightarrow{\tau_i} p_i(\mathbf{x})$ for all $i \in I$.

Proof: $(\Rightarrow)$: Follows by Prop N16.4(a), since all the projection maps p_i are continuous.

$(\Leftarrow)$: Assume $(p_i(\mathbf{x}_n)) \xrightarrow{\tau_i} p_i(\mathbf{x})$ for all $i \in I$. Let U be a basic nbhd of $\mathbf{x}$. Then U is of the form $\prod_{i \in I} U_i$, where $U_i \neq X_i$ for only finitely many i. Let $F = \{i_1, i_2, \ldots, i_k\}$ be the finite index set for which $U_i \neq X_i$. For each $i \in F$, there is (by assumption) an m_i such that $p_i(\mathbf{x}_n) \in U_i$ for all $n \geq m_i$. If we let $m = \sup_{i \in F}\{m_i\}$, then $\mathbf{x}_n \in U$ for all $n \geq m$. So $(\mathbf{x}_n)$ is eventually in every nbhd of $\mathbf{x}$, and therefore $(\mathbf{x}_n) \to \mathbf{x}$. ■

Remarks: Recalling that members in the product space X are functions on the index set I, Prop N16.5 justifies calling the product topology τ the **topology of pointwise (or componentwise) convergence**. Note that the above proof of $(\Leftarrow)$ would not work for the box topology, since the given proof requires that only finitely many of the U_i are restricted.

In a real analysis course, one typically learns that a sequence $(\mathbf{x}_k) = (x_{1k}, x_{2k}, \ldots, x_{nk})$ in $\mathbb{R}^n$ converges to $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ iff $(x_{jk}) \to x_j$ in $\mathbb{R}$ for all $j = 1, 2, \ldots, n$. This result is a special case of Prop N16.5 for finite products. In general, when one talks about pointwise convergence of a sequence of functions in analysis, the underlying topology is the product topology on the appropriate function space.

At the end of §21, Munkres defines **uniform convergence** of a sequence of functions (f_n) which map from a set X to a metric space Y. The theorem that follows (Theorem 21.6) is commonly presented in real analysis courses. Note that the functions $f_n : X \to Y$ can be interpreted as points $\mathbf{x}_n$ in the product space $\prod_{i \in X} Y = Y^X$. Munkres remarks about this after proving Theorem 21.6; he points out that the condition

$(f_n) \to (f)$ uniformly

is equivalent to the condition

$(f_n) \xrightarrow{\tau_{\bar{\rho}}} (f)$ when f_n and f are considered as elements of the metric space $(Y^X, \tau_{\bar{\rho}})$.

HW 7 (due Monday 10/17):

Prove Prop N16.1, Prop N16.2, Prop N16.3(b,c)

more to come