HOMOGENIZATION OF A VISCOELASTIC MATRIX IN LINEAR FRICTIONAL CONTACT

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ABSTRACT. The paper is devoted to study of acoustic wave propagation in a partially consolidated composite material containing loose particles. Friction of particles against the consolidated part of the material causes mechanical energy dissipation. This situation is modelled by assuming that the medium has a periodic microstructure changing rapidly on the small scale ϵ. Each of the periodic microscopic cells is composed of a viscoelastic matrix containing a rigid particle in frictional contact with the matrix. We use the methods of two-scale convergence to obtain effective acoustic equations for the homogenized material. The effective equations are history-dependent and contain the body force term, reminiscent of the well known Stokes drag force.

1. INTRODUCTION

We consider a model for the propagation of acoustic waves in a partially consolidated material, where the material properties change rapidly on a small scale characterized by a parameter ε. Our model is suggested by a work of Buckingham [4, 5, 6] where the dissipation of acoustic energy in an unconsolidated material is caused primarily by the rubbing of grains against one another.

Some mathematical results on frictional contact can be found in [10, 18, 17, 11]. The authors of these articles study the deformation of a body coming into frictional contact with an absolutely rigid foundation. It is assumed that the material properties of the body are given by the Kelvin-Voight viscoelastic constitutive equations [18, 17]. Moreover, the friction is assumed to be dry, and the friction law is given by a version of Coulomb's law

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known as normal compliance. The contact conditions of Coulomb type are formulated as inequalities involving tangential and normal forces on the contact surface.

Unfortunately, the results mentioned can not be directly extended to the case of dry friction between two elastic bodies. Analysis of the lubricated friction presents an even more challenging problem. Consequently, we consider a simplified problem where a partially consolidated medium is modelled as a poroelastic material that consists of a connected viscoelastic matrix whose pores contain rigid particles. Particles come into frictional contact with the matrix, but particle-particle contacts are prohibited. A further simplification is the assumption that the material has a periodic structure.

The microscale problem with Coulomb type contact conditions is formulated as a variational inequality. Averaging this inequality would probably lead to another variational inequality or even more general inclusion. In order to guarantee that the effective model has the form of an equation, in [14] we approximated Coulomb contact conditions by non-linear equations. Using Taylor expansions, we constructed a family of microscopic models of increasing complexity, and averaged these models using formal two-scale asymptotic expansions and homogenization.

In this paper, we present a rigorous analysis of the simplest model from [14] that corresponds to completely linearized contact conditions. The method of two-scale convergence is used to derive the effective equations. We show that the resulting effective equations have not only memory terms but also the so called drag force terms which were present in our work [14]. The drag force is the macroscopic body force, reminiscent of the classical Stokes drag force.

Homogenization in acoustics of composite materials with *consolidated* solid phase was addressed in [7], [21] by using formal expansions, and rigorously in [12], [15] for periodic geometry, and in [13] for more general irregular geometries. However, this paper seems to be first where acoustic homogenization of a non-consolidated material is treated rigorously.

The paper is organized as follows: In section 2, we describe the geometry of the medium and formulate the microscale problem (*problem 1*). Section 3 provides the proof of existence and uniqueness of solution to *problem 1* using the standard Galerkin's method. In section 4, we use the method of two scale convergence to derive the effective equation whose coefficients contain solutions of several cell problems. In section 5, we prove existence and uniqueness of solutions to these cell problems.

2. Formulation of Problem 1

Let $\mathbf{U}$ be a bounded domain in $\mathbf{R}^n$ containing a large number of identical, periodically arranged cells. First we define the unit cell $\mathbf{Y} = [0,1]^n$. Let $\mathbf{Y}_p$ (rigid particle part) be a closed subset of $\mathbf{Y}$ and $\mathbf{Y}_s = \mathbf{Y}/\mathbf{Y}_p$ be the viscoelastic solid part. Let $\mathbf{\Gamma}$ be the interface of $\mathbf{Y}_s$ and $\mathbf{Y}_r$, assumed to be smooth. For any set $\mathbf{D} \subset \mathbf{R}^n$, define $\epsilon\mathbf{D} = \{x : \epsilon^{-1}x \in \mathbf{D}\}$, and $\mathbf{D}^k = \mathbf{D} + \mathbf{k}$ for $\mathbf{k} \in \mathbf{Z}^n$. Next, define

$$\mathbf{U}_\epsilon = \cup\{\epsilon\mathbf{Y}_s^k : \epsilon\mathbf{Y}_s^k \subset \mathbf{U}, k \in \mathbf{Z}^n\}$$

$$\mathbf{\Gamma}_\epsilon = \cup\{\epsilon\mathbf{\Gamma}^k : \epsilon\mathbf{\Gamma}^k \subset \mathbf{U}, k \in \mathbf{Z}^n\}$$

Now we describe the problem that we are interested in. Let $\mathbf{u}^\epsilon(t,x)$ be the displacements of the matrix, and $\mathbf{v}^\epsilon = \dot{\mathbf{u}}^\epsilon = \partial_t\mathbf{u}$ be the velocity vector. The equation of motion is given by

$$\dot{\mathbf{v}} = \text{div }\boldsymbol{\sigma}^\epsilon + \mathbf{f} \text{ in } \mathbf{U}_\epsilon \tag{2.1}$$

We assume that the stress tensor $\boldsymbol{\sigma}^\epsilon$ satisfies Kelvin-Voight constiutive equation of linear visco-elasticity:

$$\boldsymbol{\sigma}^\epsilon = A^\epsilon e(\mathbf{u}^\epsilon) + B^\epsilon e(\mathbf{v}^\epsilon) \tag{2.2}$$

where

$$e(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

The coefficients in the constitutive equations are

$$A^\epsilon(x) = (A_{ijkl}(x, \frac{x}{\epsilon})), B^\epsilon(x) = (B_{ijkl}(x, \frac{x}{\epsilon})).$$

We assume that $A(x, y)$ and $B(x, y)$ are smooth, periodic in y and satisfy the usual symmetry

$$A_{ijkl} = A_{ijlk} = A_{jikl} = A_{klij}, \tag{2.3}$$

$$B_{ijkl} = B_{ijlk} = B_{jikl} = B_{klij}, \tag{2.4}$$

and ellipticity conditions

$$A_{ijkl} h_{ij} h_{kl} \geq \lambda_1 h_{ij} h_{ij}, \tag{2.5}$$

$$B_{ijkl} h_{ij} h_{kl} \geq \lambda_2 h_{ij} h_{ij}, \tag{2.6}$$

where λ_1 and λ_2 are positive constants independent of ϵ.

On the bondary $\partial \mathbf{U}$, we assume a homogeneous Dirichlet condition

$$\mathbf{u}^\epsilon|_{\partial \mathbf{U}} = 0 \tag{2.7}$$

To describe the contact and friction condition on $\mathbf{\Gamma}^\epsilon$, we introduce the following notation. Let $\mathbf{n} = (n_1, n_2, \cdots, n_n)$ be the unit normal on $\mathbf{\Gamma}$ pointing into $\mathbf{Y}_p$, we denote the normal and tangential components of the displacement as

$$u_n = \mathbf{u} \cdot \mathbf{n}, \qquad \mathbf{u}_T = \mathbf{u} - u_n \mathbf{n},$$

and the normal and tangential components of the stress tensor as

$$\sigma_n = \sigma_{ij} n_i n_j, \qquad \boldsymbol{\sigma}_T = \boldsymbol{\sigma} \cdot \mathbf{n} - \sigma_n \mathbf{n}.$$

We impose the following linear contact and friction condition on $\mathbf{\Gamma}_\epsilon$:

$$\sigma_n^\epsilon = -\epsilon g_\epsilon(x) - \epsilon h_\epsilon(x) u_n^\epsilon, \tag{2.8}$$

$$\boldsymbol{\sigma}_T^\epsilon = -\epsilon \mathbf{v}_T^\epsilon, \tag{2.9}$$

where $g_\epsilon(x) = g(\frac{x}{\epsilon}), h_\epsilon(x) = h(\frac{x}{\epsilon})$. We assume that $g(y)$ and $h(y)$ are smooth, positive and periodic in y.

The initial conditions are

$$\mathbf{v}^\epsilon(x,0) = 0, \mathbf{u}^\epsilon(x,0) = 0. \tag{2.10}$$

In order to give the variational formulation of the problem, let us introduce the following Banach spaces:

$$H_\epsilon = \{\mathbf{u} \in [H^1(\mathbf{U}_\epsilon)]^n, \mathbf{u}|_{\partial \mathbf{U}} = 0\}, V = [L^2(\mathbf{U}_\epsilon)]^n \tag{2.11}$$

The variational form of the problem (2.1) (2.8) and (2.9) can then be expressed as follows:

Problem 1: Find $\mathbf{u}^\epsilon \in L^\infty([0,T], H_\epsilon), \partial_t \mathbf{u} \in L^\infty([0,T], V) \cap L^2([0,T], H_\epsilon), \partial_{tt}\mathbf{u} \in L^\infty([0,T], H'_\epsilon)$ so that $\mathbf{u}^\epsilon|_{t=0} = 0, \partial_t \mathbf{u}^\epsilon|_{t=0} = 0$ and $\mathbf{u}$ satisfies

$$\begin{aligned}
&\int_{\mathbf{U}_\epsilon} \partial_{tt}\mathbf{u}^\epsilon \cdot \boldsymbol{\phi} + \int_{\mathbf{U}_\epsilon} [Ae(\mathbf{u}^\epsilon)] : e(\boldsymbol{\phi}) + \int_{\mathbf{U}_\epsilon} [Be(\partial_t \mathbf{u}^\epsilon)] : e(\boldsymbol{\phi}) \\
&+ \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon \partial_t \mathbf{u}^\epsilon \cdot \boldsymbol{\phi} - \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon(\partial_t \mathbf{u}^\epsilon \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\phi}) + \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon g_\epsilon (\mathbf{n} \cdot \boldsymbol{\phi}) + \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon h_\epsilon (\mathbf{u}^\epsilon \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\phi}) \\
&= \int_{\mathbf{U}_\epsilon} \mathbf{f} \cdot \boldsymbol{\phi} \text{ for any } \boldsymbol{\phi} \in H_\epsilon
\end{aligned} \tag{2.12}$$

where we assume $\mathbf{f} \in L^2([0,T], [L^2(U)]^n)$.

3. Existence and uniqueness of a solution to Problem 1

We shall use Galerkin's method to demonstrate that a solution to Problem 1 exists for each $\epsilon > 0$. For simplicity of notation, we suppress ϵ dependence in this section. Since H_ϵ is separable, there exists a complete system of functions $\{\boldsymbol{\phi}^k(x)\}$ in H_ϵ. For each m, we shall look for approximate solutions $\mathbf{u}_m(x,t)$ of the form

$$\mathbf{u}_m(x,t) = \sum_{l=1}^{m} g_{lm}(t) \boldsymbol{\phi}^l(x). \tag{3.1}$$

Substituting this expression into (2.12) we determine the $g_{lm}(t)$

$$\begin{aligned} &\sum_{l=1}^{m}(\int_{\mathbf{U}_\epsilon} \boldsymbol{\phi}^l \cdot \boldsymbol{\phi}^j)g''_{lm} + \sum_{l=1}^{m}\int_{\mathbf{U}_\epsilon}[Ae(\boldsymbol{\phi}^l)] : e(\boldsymbol{\phi}^j)g_{lm} + \sum_{l=1}^{m}\int_{\mathbf{U}_\epsilon}[Be(\boldsymbol{\phi}^l)] : e(\boldsymbol{\phi}^j)g'_{lm} \\ &+ \int_{\mathbf{\Gamma}_\epsilon} \epsilon\{\boldsymbol{\phi}^l \cdot \boldsymbol{\phi}^j - (\boldsymbol{\phi}^l \cdot \mathbf{n})(\boldsymbol{\phi}^j \cdot \mathbf{n})\}g'_{lm} + \epsilon\sum_{l=1}^{m}\int_{\mathbf{\Gamma}_\epsilon} h_\epsilon(\boldsymbol{\phi}^l \cdot \mathbf{n})(\boldsymbol{\phi}^j \cdot \mathbf{n})g_{lm} \\ &= \left(\int_{\mathbf{U}_\epsilon} \mathbf{f}(t)\cdot \boldsymbol{\phi}^j - \epsilon\int_{\mathbf{\Gamma}_\epsilon} g_\epsilon(\boldsymbol{\phi}^j \cdot \mathbf{n})\right), j = 1, 2, \ldots, m. \end{aligned} \tag{3.2}$$

Since the matrix $\left(\int_{\mathbf{U}_\epsilon} \boldsymbol{\phi}^l \cdot \boldsymbol{\phi}^j\right)$ is nonsingular, the above can be written as

$$g''_{jm} + \sum_{l=1}^{m}\alpha_{jl}g'_{lm} + \sum_{l=1}^{m}\beta_{jl}g_{lm} = \sum_{l=1}^{m}\gamma_{jl}\left(\int_{\mathbf{U}_\epsilon} \mathbf{f}(t)\cdot\boldsymbol{\phi}^j - \epsilon\int_{\mathbf{\Gamma}_\epsilon} g(\boldsymbol{\phi}^l \cdot \mathbf{n})\right), 1 \le j \le m, \tag{3.3}$$

where $(\alpha_{jl}), (\beta_{jl})$ and (γ_{jl}) are constant matrices.

The initial conditions for the coefficients $g_{jm}(t)$ are

$$g_{jm}(0) = g'_{jm}(0) = 0, 1 \le j \le m. \tag{3.4}$$

The system (3.3) and (3.4) has a unique solution g_{jm} on the interval [0,T]. Since $(\mathbf{f}(t), \boldsymbol{\phi}) = \int_{\mathbf{U}_\epsilon} \mathbf{f}(t)\cdot\boldsymbol{\phi}$ and $(g, \boldsymbol{\phi}) = \int_{\mathbf{\Gamma}_\epsilon} g(\boldsymbol{\phi}^l \cdot \mathbf{n})$ are square integrable with respect to t, so are the $g_{jm}, g'_{jm}, g''_{jm}$; hence,

$$\mathbf{u}_m \in L^2([0,T], H_\epsilon), \partial_t\mathbf{u}_m \in L^2([0,T], H_\epsilon), \partial_{tt}\mathbf{u}_m \in L^2([0,T], H_\epsilon) \tag{3.5}$$

By (3.2), $\mathbf{u}_m(x,t) = \sum_{l=1}^{m} g_{lm}\boldsymbol{\phi}^l$ satisfies

$$\begin{aligned} &\int_{\mathbf{U}_\epsilon} \partial_{tt}\mathbf{u}_m \cdot \boldsymbol{\phi}^j + \int_{\mathbf{U}_\epsilon}[Ae(\mathbf{u}_m)] : e(\boldsymbol{\phi}^j) + \int_{\mathbf{U}_\epsilon}[Be(\partial_t\mathbf{u}_m)] : e(\boldsymbol{\phi}^j) \\ &+ \int_{\mathbf{\Gamma}_\epsilon} \epsilon\partial_t\mathbf{u}_m \cdot \boldsymbol{\phi}^j - \int_{\mathbf{\Gamma}_\epsilon} \epsilon(\partial_t\mathbf{u}_m \cdot \mathbf{n})(\mathbf{n}\cdot\boldsymbol{\phi}^j) + \int_{\mathbf{\Gamma}_\epsilon} \epsilon g_\epsilon(\mathbf{n}\cdot\boldsymbol{\phi}^j) + \int_{\mathbf{\Gamma}_\epsilon} \epsilon h_\epsilon(\mathbf{n}\cdot\mathbf{u}_m)(\mathbf{n}\cdot\boldsymbol{\phi}^j) \\ &= \int_{\mathbf{U}_\epsilon} \mathbf{f}\cdot\cdot\boldsymbol{\phi}^j, j = 1, 2, \ldots, m. \end{aligned} \tag{3.6}$$

Therefore, we have the following Lemma:

Lemma 3.1. *There exists a solution* $\mathbf{u}_m(x,t)$ *to (3.6) such that* $\mathbf{u}_m(x,t)$ *satisfies (3.5) and* $\mathbf{u}_m(x,0) = \partial_t \mathbf{u}_m(x,0) = 0$.

We introduce the notation:

$$(\mathbf{u}, \boldsymbol{\phi}) = \int_{\mathbf{U}_\epsilon} \mathbf{u} \cdot \boldsymbol{\phi}, \qquad \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2_{L^2} = \int_{\mathbf{U}_\epsilon} |\mathbf{u}|^2. \tag{3.7}$$

Multiplying (3.6) by g'_{mj} and summing, we obtain:

(3.8)
$$\int_{\mathbf{U}_\epsilon} \partial_{tt}\mathbf{u}_m \cdot \partial_t \mathbf{u}_m + \int_{\mathbf{U}_\epsilon} [Ae(\mathbf{u}_m)] : e(\partial_t \mathbf{u}_m) + \int_{\mathbf{U}_\epsilon} [Be(\partial_t \mathbf{u}_m)] : e(\partial_t \mathbf{u}_m)$$
$$+ \int_{\mathbf{\Gamma}_\epsilon} \epsilon \partial_t \mathbf{u}_m \cdot \partial_t \mathbf{u}_m - \int_{\mathbf{\Gamma}_\epsilon} \epsilon (\partial_t \mathbf{u}_m \cdot \mathbf{n})^2 + \int_{\mathbf{\Gamma}_\epsilon} \epsilon g(\mathbf{n} \cdot \partial_t \mathbf{u}_m) + \int_{\mathbf{\Gamma}_\epsilon} \epsilon h_\epsilon (\mathbf{n} \cdot \mathbf{u}_m)(\mathbf{n} \cdot \partial_t \mathbf{u}_m)$$
$$= \int_{\mathbf{U}_\epsilon} \mathbf{f} \cdot \partial_t \mathbf{u}_m.$$

Lemma 3.2. *Let* $\mathbf{u}_m$ *be the solution indicated in Lemma 3.1, then*

$$\sup_{t \in [0,T]} \|\nabla \mathbf{u}_m(t)\|^2 \le C \left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt \right) \tag{3.9}$$

$$\sup_{t \in [0,T]} \|\partial_t \mathbf{u}_m(t)\|^2 \le C \left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt \right) \tag{3.10}$$

$$\int_0^T \|\nabla \partial_t \mathbf{u}_m(t)\|^2 dt \le C \left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt \right) \tag{3.11}$$

Proof. Let us define

$$q(t) = \|\partial_t \mathbf{u}_m\|^2 + \int_{\mathbf{U}_\epsilon} [Ae(\mathbf{u}_m)] : e(\mathbf{u}_m) + \frac{1}{2}\epsilon \int_{\mathbf{\Gamma}_\epsilon} h_\epsilon (\mathbf{u}_m \cdot \mathbf{n})^2.$$

Since

$$(\partial_t \mathbf{u} \cdot \partial_t \mathbf{u}) - (\partial_t \mathbf{u} \cdot \mathbf{n})^2 \ge 0,$$

and

$$\left| \epsilon \int_{\epsilon \mathbf{\Gamma}_\epsilon} g \mathbf{n} \cdot \partial_t \mathbf{u}_m \right| \le \frac{1}{2}\epsilon \int_{\mathbf{\Gamma}_\epsilon} g^2 + \frac{1}{2}\epsilon \int_{\mathbf{\Gamma}_\epsilon} |\partial_t \mathbf{u}_m|^2,$$

(3.8) implies

$$\frac{1}{2}\frac{dq}{dt}+\int_{\mathbf{U}_\epsilon}[Be(\partial_t\mathbf{u}_m)]:e(\partial_t\mathbf{u}_m)-\frac{1}{2}\epsilon\int_{\mathbf{\Gamma}_\epsilon}|\partial_t\mathbf{u}_m|^2\leq(\mathbf{f},\partial_t\mathbf{u}_m)+\frac{1}{2}\epsilon\int_{\mathbf{\Gamma}_\epsilon}g^2. \tag{3.12}$$

Ellipticity of B implies

$$\int_{\mathbf{U}_\epsilon}[Be(\partial_t\mathbf{u}_m)]:e(\partial_t\mathbf{u}_m)\geq K_1\|\nabla\partial_t\mathbf{u}_m\|, \tag{3.13}$$

where $K_1>0$ is a constant depending only on λ_2.

Applying trace theorem on $\mathbf{U}_\epsilon$ and rescaling we obtain

$$\frac{1}{2}\epsilon\int_{\mathbf{\Gamma}_\epsilon}\partial_t\mathbf{u}_m^2\leq K_2\left(\|\partial_t\mathbf{u}_m\|^2+\epsilon^2\|\nabla\partial_t\mathbf{u}_m\|^2\right), \tag{3.14}$$

where $K_2>0$ is a constant independent of ϵ.

For ϵ small enough, $\epsilon^2K_2\leq K_1/2$, hence

$$\begin{aligned}\int_{\mathbf{U}_\epsilon}[Be(\partial_t\mathbf{u}_m)]:e(\partial_t\mathbf{u}_m)-\frac{1}{2}\epsilon\int_{\mathbf{\Gamma}_\epsilon}\partial_t\mathbf{u}_m^2&\geq(K_1-\epsilon^2K_2)\|\nabla\partial_t\mathbf{u}_m\|^2-K_2\|\partial_t\mathbf{u}_m\|^2\\&\geq\frac{1}{2}K_1\|\nabla\partial_t\mathbf{u}_m\|^2-K_2q(t)\end{aligned} \tag{3.15}$$

Using $(\mathbf{f}(t),\partial_t\mathbf{u}_m)\leq\frac{1}{2}(\|\mathbf{f}(t)\|^2+\|\partial_t\mathbf{u}_m\|^2)$ (3.8) we have

$$\frac{dq}{dt}-(2K_2+1)q(t)+K_1\|\nabla\partial_t\mathbf{u}_m\|^2\leq\|\mathbf{f}(t)\|^2+\|g(t)\|^2_{L^2(\mathbf{\Gamma})}, \tag{3.16}$$

so that

$$q(t)\leq e^{(2K_2+1)t}\int_0^T e^{-(2K_2+1)s}\left(\|\mathbf{f}(s)\|^2+\|g(s)\|^2_{L^2(\mathbf{\Gamma})}\right)ds, \tag{3.17}$$

Integrating (3.16) with respect to t over $[0,T]$:

$$q(T)+K_1\int_0^T\|\nabla\partial_t\mathbf{u}_m\|^2dt\leq(2K_2+1)\int_0^T q(t)dt+\int_0^T\left(\|\mathbf{f}(t)\|^2+\|g(t)\|^2_{L^2(\mathbf{\Gamma})}\right)dt \tag{3.18}$$

The lemma follows from (3.17), (3.18) and the ellipticity of tensor A.

□

Lemma 3.3. *Let* $\mathbf{u}_m$ *be as in Lemma 3.1, then*

$$\sup_{t\in[0,T]} \|\mathbf{u}_m(t)\|^2 \le C\left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt\right). \tag{3.19}$$

Proof. The Lemma follows from Lemma 3.2 and $\mathbf{u}_m = \int_0^t \partial_t \mathbf{u}_m dt$.

□

Lemma 3.4. *There exists a weak solution* $\mathbf{u} \in L^\infty([0,T], H_\epsilon)$ *to Problem 1.*

Proof. By Lemma 3.2 and Lemma 3.3,

$$\{\mathbf{u}_m\} \text{ is bounded in } L^\infty([0,T], H_\epsilon), \tag{3.20}$$

hence there exists an element $\mathbf{u} \in L^\infty([0,T], H_\epsilon)$ such that for a subsequence which we still denote by $\{\mathbf{u}_m\}$

$$\mathbf{u}_m \text{ converges to } \mathbf{u} \text{ in the weak* topology of } L^\infty([0,T], H_\epsilon), \tag{3.21}$$

that is

$$\int_0^T (\mathbf{u}_m(t) - \mathbf{u}(t), \boldsymbol{\phi}(t))_{H_\epsilon} dt \to 0 \text{ for any } \boldsymbol{\phi} \in L^1([0,T], H_\epsilon). \tag{3.22}$$

Let γ be the trace operator on $\mathbf{\Gamma}_\epsilon$. By the trace theorem (see (3.14)),

$$\{\gamma\mathbf{u}_m\} \text{ is bounded in } L^\infty([0,T], [L^2(\mathbf{\Gamma}_\epsilon)]^n), \tag{3.23}$$

hence there exists an element $\mathbf{u}^* \in L^\infty([0,T], [L^2(\mathbf{\Gamma}_\epsilon)]^n)$ such that for a subsequence which we still denote by $\{\mathbf{u}_m\}$

$$\gamma\mathbf{u}_m \text{ converges to } \mathbf{u}^* \text{ in the weak* topology of } L^\infty([0,T], [L^2(\mathbf{\Gamma}_\epsilon)]^n), \tag{3.24}$$

that is

$$\int_0^T (\gamma\mathbf{u}_m(t) - \mathbf{u}^*(t), \boldsymbol{\phi}(t))_{L^2(\mathbf{\Gamma}_\epsilon)} dt \to 0 \text{ for any } \boldsymbol{\phi} \in L^1([0,T], [L^2(\mathbf{\Gamma}_\epsilon)]^n). \tag{3.25}$$

For $\psi \in C[0,T], \boldsymbol{\phi}(x) \in [C^\infty(\mathbf{U}_\epsilon)]^n$, using integration by parts, we obtain

$$\int_{\mathbf{U}_\epsilon} \nabla \mathbf{u}_m \cdot \boldsymbol{\phi} = -\int_{\mathbf{U}_\epsilon} \mathbf{u}_m \cdot \nabla \boldsymbol{\phi} + \int_{\boldsymbol{\Gamma}_\epsilon} \gamma \mathbf{u}_m (\boldsymbol{\phi} \cdot \mathbf{n}). \tag{3.26}$$

Multiplying the above by ψ and integrating we have

$$\int_0^T \psi dt \int_{\mathbf{U}_\epsilon} \nabla \mathbf{u}_m \cdot \boldsymbol{\phi} = -\int_0^T \psi dt \int_{\mathbf{U}_\epsilon} \mathbf{u}_m \cdot \nabla \boldsymbol{\phi} + \int_0^T \psi dt \int_{\boldsymbol{\Gamma}_\epsilon} \gamma \mathbf{u}_m (\boldsymbol{\phi} \cdot \mathbf{n}), \tag{3.27}$$

and letting $m \to \infty$ in above equation yields

$$\int_0^T \psi dt \int_{\mathbf{U}_\epsilon} \nabla \mathbf{u} \cdot \boldsymbol{\phi} = -\int_0^T \psi dt \int_{\mathbf{U}_\epsilon} \mathbf{u} \cdot \nabla \boldsymbol{\phi} + \int_0^T \psi dt \int_{\boldsymbol{\Gamma}_\epsilon} \mathbf{u}^* (\boldsymbol{\phi} \cdot \mathbf{n}). \tag{3.28}$$

Integrating by parts once more we have

$$\int_0^T \psi dt \int_{\boldsymbol{\Gamma}_\epsilon} (\mathbf{u}^* - \gamma \mathbf{u})(\boldsymbol{\phi} \cdot \mathbf{n}) = 0 \text{ for all } \psi \text{ and } \boldsymbol{\phi}\ ; \tag{3.29}$$

it follows that

$$\mathbf{u}^* = \gamma \mathbf{u}. \tag{3.30}$$

In order to pass the limit in (3.6), let us consider a scalar function $\psi(t) \in C^2[0,T]$ satisfying $\psi(T) = \psi'(T) = 0$. Multiplying (3.6) by ψ and integrating by parts we obtain

(3.31)

$$\begin{aligned}
&\int_0^T \psi''(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{u}_m \cdot \boldsymbol{\phi}^j + \int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} [Ae(\mathbf{u}_m)] : e(\boldsymbol{\phi}^j) \\
&- \int_0^T \psi'(t)dt \int_{\mathbf{U}_\epsilon} [Be(\mathbf{u}_m)] : e(\boldsymbol{\phi}^j) - \int_0^T \psi'(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon \mathbf{u}_m \cdot \boldsymbol{\phi}^j + \int_0^T \psi'(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon (\mathbf{u}_m \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\phi}^j) \\
&+ \int_0^T \psi(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon g(\mathbf{n} \cdot \boldsymbol{\phi}^j) + \int_0^T \psi(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon h_\epsilon (\mathbf{n} \cdot \mathbf{u}_m)(\mathbf{n} \cdot \boldsymbol{\phi}^j) \\
&= \int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{f} \cdot \boldsymbol{\phi}^j, j = 1, 2, \ldots, m
\end{aligned}$$

Letting $m \to \infty$ in the above equation results in

$$
\begin{aligned}
&\int_0^T \psi''(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{u}\cdot\boldsymbol{\phi}^j + \int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} [Ae(\mathbf{u})] : e(\boldsymbol{\phi}^j) \\
&- \int_0^T \psi'(t)dt \int_{\mathbf{U}_\epsilon} [Be(\mathbf{u})] : e(\boldsymbol{\phi}^j) - \int_0^T \psi'(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon\mathbf{u}\cdot\boldsymbol{\phi}^j + \int_0^T \psi'(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon(\mathbf{u}\cdot\mathbf{n})(\mathbf{n}\cdot\boldsymbol{\phi}^j) \\
&+ \int_0^T \psi(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon g(\mathbf{n}\cdot\boldsymbol{\phi}^j) + \int_0^T \psi(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon h_\epsilon(\mathbf{n}\cdot\mathbf{u})(\mathbf{n}\cdot\boldsymbol{\phi}^j) \\
&= \int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{f}\cdot\boldsymbol{\phi}^j, j = 1, 2\ldots
\end{aligned}
\tag{3.32}
$$

Clearly as (3.32) is a weak form of Problem 1, we obtain the lemma.

□

Theorem 3.5. *There exists a unique solution* $\mathbf{u} \in L^\infty([0,T], H_\epsilon), \partial_t\mathbf{u} \in L^\infty([0,T], V) \cap L^2([0,T], H_\epsilon), \partial_{tt}u \in L^\infty([0,T], H'_\epsilon)$ *to Problem 1. Furthermore* $\mathbf{u}$ *satisfies the estimates*

$$\sup_{t\in[0,T]} \|\mathbf{u}(t)\|^2_{H_\epsilon} \le C\left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt\right) \tag{3.33}$$

$$\sup_{t\in[0,T]} \|\partial_t\mathbf{u}(t)\|^2 \le C\left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt\right) \tag{3.34}$$

$$\int_0^T \|\nabla\partial_t\mathbf{u}(t)\|^2 dt \le C\left(\int_0^T \|\mathbf{f}(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\mathbf{\Gamma})} dt\right) \tag{3.35}$$

where C *is positive constant independent of* ϵ.

Proof. By Lemma 3.2,

$$\text{the sequence } \{\partial_t\mathbf{u}_m\} \text{ is bounded in } L^2([0,T], H_\epsilon); \tag{3.36}$$

hence, there exists a subsequence, which we also denote by $\{\partial_t \mathbf{u}_m\}$, that converges weakly to some $\mathbf{v} \in L^2([0,T], H_\epsilon)$. Furthermore,

$$\text{the sequence } \{\partial_t \mathbf{u}_m\} \text{ converges to } \mathbf{v} \text{ for the weak topology of } L^2([0,T], H_\epsilon), \tag{3.37}$$

so for any $\boldsymbol{\phi} \in C^1([0,T], H_\epsilon)$, we have

$$\lim_{m\to\infty} \int_0^T (\partial_t \mathbf{u}_m, \boldsymbol{\phi}) dt = \int_0^T (\mathbf{v}, \boldsymbol{\phi}) dt. \tag{3.38}$$

On the other hand, using integration by parts we find

$$\lim_{m\to\infty} \int_0^T (\partial_t \mathbf{u}_m, \boldsymbol{\phi}) dt = -\lim_{m\to\infty} \int_0^T (\mathbf{u}_m, \partial_t \boldsymbol{\phi}) dt = -\int_0^T (\mathbf{u}, \partial_t \boldsymbol{\phi}) dt; \tag{3.39}$$

hence,

$$\int_0^T (\mathbf{v}, \boldsymbol{\phi}) dt = -\int_0^T (\mathbf{u}, \partial_t \boldsymbol{\phi}) dt. \tag{3.40}$$

Therefore,

$$\mathbf{v} = \partial_t \mathbf{u}. \tag{3.41}$$

Also, by Lemma 3.2,

$$\text{the sequence } \{\partial_t \mathbf{u}_m\} \text{ is bounded in } L^\infty([0,T], V); \tag{3.42}$$

hence, there exists a subsequence, which we also denote by $\{\partial_t \mathbf{u}_m\}$, which converges in weak-* topology to some $\mathbf{v}^* \in L^\infty([0,T], V)$. Steps similar to (3.38) -(3.41) lead to $\partial_t \mathbf{u} = \mathbf{v}^*$, so we have $\partial_t \mathbf{u} \in L^\infty([0,T], V)$.

Since $\mathbf{u} \in L^\infty([0,T], H_\epsilon)$, $\partial_t \mathbf{u} \in L^\infty([0,T], V)$, it follows that $Ae(\mathbf{u}) \in L^\infty([0,T], V)$, $Be(\partial_t \mathbf{u}) \in L^\infty([0,T], V)$, and we conclude from (2.12) that $\partial_{tt} \mathbf{u} \in L^\infty([0,T], H'_\epsilon)$.

Using the same steps as in Lemma 3.2, we obtain (3.33)-(3.35).

□

4. Convergence and Effective equations

4.1. **Two-scale convergence.** We are going to use the notion of two-scale convergence which was introduced by Nguetseng [20] and

Allaire [2] . Let $C^\infty_\#(\mathbf{Y})$ denote the space of those C^∞ functions periodic on $\mathbf{Y}$.

Definition 4.1. $\{\mathbf{u}^\epsilon(x,t)\} \subset [L^2([0,T]\times\mathbf{U})]^n$ two-scale converges to $\mathbf{u}(t,x,y) \in [L^2([0,T]\times \mathbf{U}\times\mathbf{Y})]^n$ iff for any $\boldsymbol{\phi}(t,x,y) \in [C^\infty([0,T]\times\mathbf{U}, C^\infty_\#(\mathbf{Y}))]^n$,

$$\lim_{\epsilon\to 0}\int_0^T\int_{\mathbf{U}} \mathbf{u}^\epsilon(t,x)\cdot\boldsymbol{\phi}(t,x,\frac{x}{\epsilon})dxdt = \int_0^T\int_{\mathbf{U}}\int_{\mathbf{Y}} \mathbf{u}(t,x,y)\cdot\boldsymbol{\phi}(t,x,y)dtdxdy. \tag{4.1}$$

One can prove the following theorem (see [20],[2]):

Theorem 4.2. *(i) If $\{\mathbf{u}^\epsilon(x,t)\}$ is a bounded sequence in $[L^2([0,T],L^2(\mathbf{U}))]^n$, then there exists $\mathbf{u}_0(t,x,y)\in[L^2([0,T]\times\mathbf{U}, L^2\#(\mathbf{Y}))]^n$ such that a subsequence of $\{\mathbf{u}^\epsilon(t,x)\}$ two-scale converges to $\mathbf{u}_0(t,x,y)$ in the sense of Definition 4.1.*
(ii)If $\{\mathbf{u}^\epsilon(x,t)\}$ is a bounded sequence in $[L^2([0,T],H^1(\mathbf{U}))]^n$, then there exist $\mathbf{u}_0(t,x)\in[L^2([0,T],H^1(\mathbf{U}))]^n$ and $\mathbf{u}_1(t,x,y)\in[L^2([0,T]\times\mathbf{U},H^1\#(\mathbf{Y}))]^n$ such that a subsequence of $\{\mathbf{u}^\epsilon(t,x)\}$ two-scale converges to $\mathbf{u}_0(t,x)$ and a subsequence of $\nabla_x\mathbf{u}^\epsilon$ two-scale converges to $\nabla_x\mathbf{u}_0+\nabla_y\mathbf{u}_1$ in the sense of Definition 4.1 .
(iii)If $\{\mathbf{u}^\epsilon(x,t)\}$ and $\{\epsilon\nabla_x\mathbf{u}^\epsilon(x,t)\}$ are bounded sequence in $[L^2([0,T],L^2(\mathbf{U}))]^n$, then there exists $\mathbf{u}_0(t,x,y)\in[L^2([0,T]\times\mathbf{U},H^1\#(\mathbf{Y}))]^n$ such that a subsequence of $\{\mathbf{u}^\epsilon(t,x)\}$ and $\{\epsilon\nabla_x\mathbf{u}^\epsilon(x,t)\}$ two-scale converges to $\mathbf{u}_0(t,x,y)$ and $\nabla_y\mathbf{u}_0(t,x,y)$ in the sense of Definition 4.1.

Proof. The proof is a simple adaption of that in [20] and [2].

□

Allaire *et al.* [3] and Neuss-Radu [19] extend two-scale convergence to periodic hypersurfaces.

Definition 4.3. $\{\mathbf{u}^\epsilon(t,x)\} \subset [L^2([0,T]\times\mathbf{\Gamma}_\epsilon)]^n$ two-scale converges to $\mathbf{u}(t,x,y) \in [L^2([0,T]\times \mathbf{U}, L^2(\mathbf{\Gamma}))]^n$ iff for any $\boldsymbol{\phi}(t,x,y) \in [C^\infty([0,T]\times\mathbf{U}, C^\infty_\#(\mathbf{Y}))]^n$,

$$\lim_{\epsilon\to 0} \epsilon \int_0^T \int_{\mathbf{\Gamma}_\epsilon} \mathbf{u}^\epsilon(t,x)\cdot\boldsymbol{\phi}(t,x,\frac{x}{\epsilon})d\sigma_\epsilon dt = \int_0^T\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \mathbf{u}(t,x,y)\cdot\boldsymbol{\phi}(t,x,y)dtdxd\sigma_y. \tag{4.2}$$

One can prove:

Theorem 4.4. *Let $\{\mathbf{u}^\epsilon(x,t)\}$ be a sequence in $[L^2([0,T],L^2(\mathbf{\Gamma}_\epsilon))]^n$ such that*

$$\epsilon\int_{\mathbf{\Gamma}_\epsilon} |\mathbf{u}^\epsilon(x)|^2 \le C$$

where C is a positive constant independent of ϵ. Then there exists $\mathbf{u}_0(t,x,y) \in [L^2([0,T]\times\mathbf{U}, L^2\#(\mathbf{\Gamma}))]^n$ such that a subsequence of $\{\mathbf{u}^\epsilon(t,x)\}$ two-scale converges to $\mathbf{u}_0(t,x,y)$ in the sense of Definition 4.3.

Proof. The proof is a simple adaption of that in [19] and [3]. □

Two scale convergence can handle homogenization problem on perforated domains conveniently without requiring any sophisticated extensions such as used in [1]. We only need to use the trivial extension by zero in the hole $\mathbf{Y}_p$ in the following lemma.

Lemma 4.5. *Let $\{\mathbf{u}^\epsilon(x,t)\} \subset L^2([0,T],H_\epsilon)$ be as in Theorem 3.5, denote by $\tilde{\ }$ the extension by zero in $\mathbf{U}-\mathbf{U}_\epsilon$then*

(i) there exists $\mathbf{u}_0(t,x) \in [L^2([0,T],H^1(\mathbf{U})]^n$ so that up to a subsequence, $\tilde{\mathbf{u}}^\epsilon(t,x)$ two-scale converges to $\mathbf{u}_0(t,x)\chi(y)$ in the sense of both Definition 4.1 and Definition 4.3. More precisely,

$$\lim_{\epsilon\to 0} \int_0^T \int_{\mathbf{U}_\epsilon} \mathbf{u}^\epsilon(t,x)\cdot\boldsymbol{\phi}(t,x,\frac{x}{\epsilon})dxdt = \int_0^T\int_{\mathbf{U}}\int_{\mathbf{Y}_s} \mathbf{u}_0(t,x)\cdot\boldsymbol{\phi}(t,x,y)dtdxdy, \tag{4.3}$$

$$\lim_{\epsilon\to 0} \epsilon \int_0^T \int_{\mathbf{\Gamma}_\epsilon} \mathbf{u}^\epsilon(t,x)\cdot\boldsymbol{\phi}(t,x,\frac{x}{\epsilon})d\sigma_\epsilon dt = \int_0^T\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \mathbf{u}_0(t,x)\cdot\boldsymbol{\phi}(t,x,y)dtdxd\sigma_y, \tag{4.4}$$

for any $\boldsymbol{\phi}(t,x,y) \in [C^\infty([0,T]\times\mathbf{U}, C^\infty_\#(\mathbf{Y}))]^n$, where $\chi(y)$ is the characteristic function on $\mathbf{Y}_s$.

(ii) There exists $\mathbf{u}_1(t,x,y) \in [L^2([0,T] \times \mathbf{U}, H^1_{\#}(\mathbf{Y}_s))]^n$ *such that a subsequence of extension of* $\nabla_x \mathbf{u}^\epsilon$ *two-scale converges to* $\chi(y)[\nabla_x \mathbf{u}_0 + \nabla_y \mathbf{u}_1]$.

Proof. Using the same steps as in the proof of Thm 2.9 in [2], we get (4.3) and part (ii). (4.4) follows from proposition 2.6 in [3] .

□

Now we rewrite (3.29) as follows.

(4.5)

$$\begin{aligned}
&\int_0^T \psi''(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{u}^\epsilon \cdot \boldsymbol{\phi} + \int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} [A^\epsilon e(\mathbf{u}^\epsilon)] : e(\boldsymbol{\phi}) \\
&- \int_0^T \psi'(t)dt \int_{\mathbf{U}_\epsilon} [B^\epsilon e(\mathbf{u}^\epsilon)] : e(\boldsymbol{\phi}) - \int_0^T \psi'(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon \mathbf{u}^\epsilon \cdot \boldsymbol{\phi} + \int_0^T \psi'(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon(\mathbf{u}^\epsilon \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\phi} \\
&+ \int_0^T \psi(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon g_\epsilon(\mathbf{n} \cdot \boldsymbol{\phi}) + \int_0^T \psi(t)dt \int_{\boldsymbol{\Gamma}_\epsilon} \epsilon h_\epsilon(\mathbf{n} \cdot \mathbf{u}^\epsilon)(\mathbf{n} \cdot \boldsymbol{\phi}) \\
&= \int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{f} \cdot \boldsymbol{\phi}.
\end{aligned}$$

If we replace the test function by $\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})$, then applying lemma 4.5, we obtain

$$\int_0^T \psi''(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{u}^\epsilon \cdot [\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})] \to \int_0^T \psi''(t)dt \int_{\mathbf{U}} \mathbf{u}_0 \cdot \boldsymbol{\phi}, \tag{4.6}$$

$$\begin{aligned}
&\int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} [A^\epsilon e_x(\mathbf{u}^\epsilon)] : e_x([\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})]) \\
&\qquad \to \int_0^T \psi(t)dt \int_{\mathbf{U}} \int_{\mathbf{Y}_s} A[e_x(\mathbf{u}_0) + e_y(\mathbf{u}_1)] : (e_x(\boldsymbol{\phi}) + e_y(\boldsymbol{\phi}_1)),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\int_0^T \psi'(t)dt \int_{\mathbf{U}_\epsilon} [B^\epsilon e_x(\mathbf{u}^\epsilon)] : e_x([\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})]) \to \\
&\qquad \int_0^T \psi'(t)dt \int_{\mathbf{U}} \int_{\mathbf{Y}_s} B[e_x(\mathbf{u}_0) + e_y(\mathbf{u}_1)] : (e_x(\boldsymbol{\phi}) + e_y(\boldsymbol{\phi}_1)),
\end{aligned} \tag{4.8}$$

$$\int_0^T \psi'(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon \mathbf{u}^\epsilon \cdot [\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})] \to \int_0^T \psi'(t)dt \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} (\mathbf{u}_0 \cdot \boldsymbol{\phi}), \tag{4.9}$$

$$\int_0^T \psi'(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon (\mathbf{u}^\epsilon \cdot \mathbf{n})(\mathbf{n} \cdot [\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})]) \to \int_0^T \psi'(t)dt \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} (\mathbf{u}_0 \cdot \mathbf{n})(\boldsymbol{\phi} \cdot \mathbf{n}), \tag{4.10}$$

$$\int_0^T \psi(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon g_\epsilon (\mathbf{n} \cdot [\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})]) \to \int_0^T \psi(t)dt \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} g(\boldsymbol{\phi} \cdot \mathbf{n}), \tag{4.11}$$

$$\int_0^T \psi(t)dt \int_{\mathbf{\Gamma}_\epsilon} \epsilon h_\epsilon (\mathbf{n} \cdot \mathbf{u}^\epsilon)(\mathbf{n} \cdot [\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})]) \to \int_0^T \psi(t)dt \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} h(\mathbf{u}_0 \cdot \mathbf{n})(\boldsymbol{\phi} \cdot \mathbf{n}), \tag{4.12}$$

$$\int_0^T \psi(t)dt \int_{\mathbf{U}_\epsilon} \mathbf{f} \cdot [\boldsymbol{\phi}(x) + \epsilon \boldsymbol{\phi}_1(x, \frac{x}{\epsilon})] \to \int_0^T \psi(t)dt \int_{\mathbf{U}} \mathbf{f} \cdot \boldsymbol{\phi}. \tag{4.13}$$

Lemma 4.6. *Let $\{\mathbf{u}_0(t,x), \mathbf{u}_1(t,x,y)\}$ be as in Lemma 4.5. Then $\{\mathbf{u}_0(t,x), \mathbf{u}_1(t,x,y)\}$satisfy the following equations in distributional sense.*

(4.14)
$$\begin{aligned} &\frac{d^2}{dt^2} \int_{\mathbf{U}} \mathbf{u}_0 \boldsymbol{\phi} + \int_{\mathbf{U}} \int_{\mathbf{Y}_s} A[e_x(\mathbf{u}_0) + e_y(\mathbf{u}_1)] : e_x(\boldsymbol{\phi}) + \frac{d}{dt} \int_{\mathbf{U}} \int_{\mathbf{Y}_s} B[e_x(\mathbf{u}_0) + e_y(\mathbf{u}_1)] : e_x(\boldsymbol{\phi}) \\ &+ \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} \partial_t \mathbf{u}_0 \cdot \boldsymbol{\phi} - \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} \partial_t \mathbf{u}_0 \cdot \mathbf{n}(\boldsymbol{\phi} \cdot \mathbf{n}) + \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} g \boldsymbol{\phi} \cdot \mathbf{n} + \int_{\mathbf{U}} \int_{\mathbf{\Gamma}} h(\mathbf{u}_0 \cdot \mathbf{n})(\boldsymbol{\phi} \cdot \mathbf{n}) \\ &= \int_{\mathbf{U}} \mathbf{f} \cdot \boldsymbol{\phi} \end{aligned}$$

$$\int_{\mathbf{U}} \int_{\mathbf{Y}_s} A[e_x(\mathbf{u}_0) + e_y(\mathbf{u}_1)] : e_y(\boldsymbol{\phi}_1) + \int_{\mathbf{U}} \int_{\mathbf{Y}_s} B[e_x(\partial_t \mathbf{u}_0) + e_y(\partial_t \mathbf{u}_1)] : e_y(\boldsymbol{\phi}_1) = 0 \tag{4.15}$$

$$\mathbf{u}_0(0) = \partial_t \mathbf{u}_0(0) = 0, \mathbf{u}_1(0) = 0 \tag{4.16}$$

Proof. Let $\epsilon \to 0$ in (4.5) and use (4.6)-(4.13):

(4.17)
$$\int_0^T \psi''(t)\int_{\mathbf{U}} \mathbf{u}_0\boldsymbol{\phi} + \int_0^T \psi(t)\int_{\mathbf{U}}\int_{\mathbf{Y}_s} A[e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1)] : e_x(\boldsymbol{\phi})$$
$$-\int_0^T \psi'(t)\int_{\mathbf{U}}\int_{\mathbf{Y}_s} B[e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1)] : e_x(\boldsymbol{\phi}) + \int_0^T \psi\int_{\mathbf{U}}\int_{\mathbf{Y}_s} A[e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1)] : e_y(\boldsymbol{\phi}_1)$$
$$-\int_0^T \psi'\int_{\mathbf{U}}\int_{\mathbf{Y}_s} B[e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1)] : e_y(\boldsymbol{\phi}_1) - \int_0^T \psi'(t)\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \mathbf{u}_0\cdot\boldsymbol{\phi}$$
$$+\int_0^T \psi'(t)\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \mathbf{u}_0\cdot\mathbf{n}(\boldsymbol{\phi}\cdot\mathbf{n}) + \int_0^T \psi(t)\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} g\boldsymbol{\phi}\cdot\mathbf{n} = \int_{\mathbf{U}} \mathbf{f}\cdot\boldsymbol{\phi}.$$

We are now able to establish the lemma by setting $\boldsymbol{\phi}_1 = 0$ and $\boldsymbol{\phi} = 0$ respectively. □

Lemma 4.7. *The system (4.14)-(4.16) has a unique solution.*

Proof. It is sufficient to prove that for $\mathbf{f} = \mathbf{0}$, $g = 0$ we have only the trivial solution $\mathbf{u}_0 = \mathbf{u}_1 = 0$. Set $\boldsymbol{\phi} = \partial_t\mathbf{u}_0, \boldsymbol{\phi}_1 = \partial_t\mathbf{u}_1$ as test function in(4.14) and (4.15):

(4.18)
$$\int_{\mathbf{U}} \partial_{tt}\mathbf{u}_0\partial_t\mathbf{u}_0 + \int_{\mathbf{U}}\int_{\mathbf{Y}_s} A(e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1)) : e_x(\partial_t\mathbf{u}_0) + \frac{d}{dt}\int_{\mathbf{U}}\int_{\mathbf{Y}_s} B(e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1)) : e_x(\partial_t\mathbf{u}_0)$$
$$+\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \partial_t\mathbf{u}_0\cdot\partial_t\mathbf{u}_0 - \int_{\mathbf{U}}\int_{\mathbf{\Gamma}}(\partial_t\mathbf{u}_0\cdot\mathbf{n})^2 + \int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \epsilon h(\mathbf{u}_0\cdot\mathbf{n})(\partial_t\mathbf{u}_0\cdot\mathbf{n}) = 0$$

$$\int_{\mathbf{U}}\int_{\mathbf{Y}_s} A(e_x(\mathbf{u}_0)+e_y(\mathbf{u}_1) : e_y(\partial_t\mathbf{u}_1) + \int_{\mathbf{U}}\int_{\mathbf{Y}_s} B(e_x(\partial_t\mathbf{u}_0)+e_y(\partial_t\mathbf{u}_1) : e_y(\partial_t\mathbf{u}_1) = 0 \tag{4.19}$$

Adding (4.18) and (4.19) and integrating in time we obtain

(4.20)
$$\int_{\mathbf{U}}(\partial_t\mathbf{u}_0)^2 + \int_{\mathbf{U}}\int_{\mathbf{Y}_s} Ae_x(\mathbf{u}_0) : e_x(\mathbf{u}_0) + 2\int_0^t\int_{\mathbf{U}}\int_{\mathbf{Y}_s} B(e_x(\partial_t\mathbf{u}_0)+e_y(\partial_t\mathbf{u}_1)) : (e_x(\partial_t\mathbf{u}_0)+e_y(\partial_t\mathbf{u}_1))$$
$$+\int_0^t\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \partial_t\mathbf{u}_0\cdot\partial_t\mathbf{u}_0 - \int_0^t\int_{\mathbf{U}}\int_{\mathbf{\Gamma}}(\partial_t\mathbf{u}_0\cdot\mathbf{n})^2 + \frac{1}{2}\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \epsilon h(\mathbf{u}_0\cdot\mathbf{n})^2 = 0.$$

Note that
$$\int_0^t\int_{\mathbf{U}}\int_{\mathbf{\Gamma}} \partial_t\mathbf{u}_0\cdot\partial_t\mathbf{u}_0 - \int_0^t\int_{\mathbf{U}}\int_{\mathbf{\Gamma}}(\partial_t\mathbf{u}_0\cdot\mathbf{n})^2 \geq 0,$$
so that (4.20) implies $\partial_t\mathbf{u}_0 = 0, \partial_t\mathbf{u}_1 = 0$, using initial condition, we have $\mathbf{u}_0 \equiv 0, \mathbf{u}_1 \equiv 0$. □

4.2. **Derivation of the effective equation of $\mathbf{u}_0$.** Since system (4.14)-(4.16) has a unique solution, we seek $\mathbf{u}_1(t,x,y)$ in the form

$$\mathbf{u}_1(t,x,y)=\sum_{ij}\Big\{\mathbf{M}^{ij}(y)e_x(\mathbf{u}_0)_{ij}(t,x)+\int_0^t\mathbf{K}^{ij}(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)ds\Big\}, \tag{4.21}$$

where the vectors $\mathbf{M}^{ij},\mathbf{K}^{ij}$ are to be specified. To derive equations for $\mathbf{M}^{ij}(y),\mathbf{K}^{ij}(y,t)$, we substitute (4.21) into (4.15):

$$\begin{aligned}&\int_{\mathbf{U}}\int_{\mathbf{Y}_s}A\Big\{e_x(\mathbf{u}_0)+\sum_{ij}\{e_y(\mathbf{M}^{ij})(y)e_x(\mathbf{u}_0)_{ij}(t,x)\\&+\int_0^t e_y(\mathbf{K}^{ij})(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)ds\}\Big\}:e_y(\boldsymbol{\phi}_1)\\&+\int_{\mathbf{U}}\int_{\mathbf{Y}_s}B\Big\{e_x(\partial_t\mathbf{u}_0)+\sum_{ij}\{e_y(\mathbf{M}^{ij})(y)e_x(\partial_t\mathbf{u}_0)_{ij}(t,x)+e_y(\mathbf{K}^{ij})(y,0)e_x(\mathbf{u}_0)_{ij}\\&+\int_0^t e_y(\partial_t\mathbf{K}^{ij})(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)ds\}\Big\}:e_y(\boldsymbol{\phi}_1)=0.\end{aligned} \tag{4.22}$$

Collecting $e_x(\partial_t\mathbf{u}_0)$ terms, and choosing $\mathbf{M}^{ij}$ so that

$$\int_{\mathbf{U}}\int_{\mathbf{Y}_s}B\left(e_x(\partial_t\mathbf{u}_0)+\sum_{ij}e_y(\mathbf{M}^{ij})(y)e_x(\partial_t\mathbf{u}_0)_{ij}(t,x)\right):e_y(\boldsymbol{\phi}_1)=0, \tag{4.23}$$

we have

$$\int_{\mathbf{Y}_s}B\Big\{\frac{1}{2}(e_i\otimes e_j+e_j\otimes e_i)+e_y(\mathbf{M}^{ij})\Big\}:e_y(\boldsymbol{\phi}_1)=0 \text{ for any } \boldsymbol{\phi}_1\in[H^1_{\#}(\mathbf{Y})]^n, \tag{4.24}$$

where $\{e_i\}$ are basis vector in $\mathbf{R}^n$.

i.e $\mathbf{M}^{ij}$ is determined by the following linear elastic system:

$$\begin{aligned}&\text{div}_y\Big\{B\left(\frac{1}{2}(e_i\otimes e_j+e_j\otimes e_i)+e_y(\mathbf{M}^{ij})\right)\Big\}=0,\\&B\left(\frac{1}{2}(e_i\otimes e_j+e_j\otimes e_i)+e_y(\mathbf{M}^{ij})\right)\cdot\mathbf{n}|=0 \text{ on } \boldsymbol{\Gamma},\\&\mathbf{M}^{ij}(y)\in[H^1_{\#}(\mathbf{Y}_s)]^n.\end{aligned} \tag{4.25}$$

Collecting $e_x(\mathbf{u}_0)$ terms in (4.22) and choosing $\mathbf{K}^{ij}(0)$ so that

$$\int_{\mathbf{U}}\int_{\mathbf{Y}_s}\Big\{Ae_x(\mathbf{u}_0)+A\sum_{ij}e_y(\mathbf{M}^{ij})(y)e_x(\mathbf{u}_0)_{ij}(t,x) + B\sum_{ij}e_y(\mathbf{K}^{ij})(0)e_x(\mathbf{u}_0)_{ij}\Big\}:e_y(\boldsymbol{\phi}_1)=0 \tag{4.26}$$

we have

(4.27)

$$\int_{\mathbf{Y}_s}\Big\{A\frac{(e_i\otimes e_j+e_j\otimes e_i)}{2}+Ae_y(\mathbf{M}^{ij})+Be_y(\mathbf{K}^{ij}(0))\Big\}:e_y(\boldsymbol{\phi}_1)=0 \text{ for any } \boldsymbol{\phi}_1\in[H^1_\#(\mathbf{Y})]^n$$

i.e $\mathbf{K}^{ij}(0)$ is determined by the following linear elastic system:

$$\begin{aligned} &\mathrm{div}_y\Big\{A\Big(\frac{1}{2}(e_i\otimes e_j+e_j\otimes e_i)\Big)+Ae_y(\mathbf{M}^{ij})+Be_y(\mathbf{K}^{ij}(0)\Big\}=0\\ &\Big\{A\Big(\frac{1}{2}(e_i\otimes e_j+e_j\otimes e_i)\Big)+Ae_y(\mathbf{M}^{ij})+Be_y(\mathbf{K}^{ij}(0)\Big\}\cdot\mathbf{n}=0 \text{ on } \boldsymbol{\Gamma},\\ &\mathbf{K}^{ij}(0,y)\in[H^1_\#(\mathbf{Y})]^n, \mathbf{M}^{ij}(y) \text{ is given by (4.26).}\end{aligned} \tag{4.28}$$

Collecting terms containing convolutions in time in (4.22) and choosing $\mathbf{K}^{ij}(t,y)$ we obtain

$$\begin{aligned}&\int_{\mathbf{U}}\int_{\mathbf{Y}_s}\int_0^t\sum_{ij}\Big\{Ae_y(\mathbf{K}^{ij})(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)\\ &+Be_y(\partial_t\mathbf{K}^{ij})(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)ds\Big\}:e_y(\boldsymbol{\phi}_1)=0,\end{aligned} \tag{4.29}$$

which implies

$$\int_{\mathbf{Y}_s}\big[Ae_y(\mathbf{K}^{ij})(y,t)+Be_y(\partial_t\mathbf{K}^{ij})(y,t)\big]:e_y(\boldsymbol{\phi}_1)=0, \tag{4.30}$$

i.e $\mathbf{K}^{ij}(t,y)$ is determined by the following system

$$
\begin{aligned}
&\operatorname{div}{}_y\Big\{Be_y(\partial_t\mathbf{K}^{ij})+Ae_y(\mathbf{K}^{ij})\Big\}=0,\\
&\Big\{Be_y(\partial_t\mathbf{K}^{ij})+Ae_y(\mathbf{K}^{ij})\Big\}\cdot\mathbf{n}=0 \text{ on } \mathbf{\Gamma},\\
&\mathbf{K}^{ij}(t,y)\in L^2([0,T],[H^1_\#(\mathbf{Y})]^n),\\
&\mathbf{K}^{ij}(0,y) \text{ is given by (4.28).}
\end{aligned}
\tag{4.31}
$$

The well-posedness of the above cell problems will be studied in next section. Assuming fro now that the cell problems are solvable, we proceed to derive the effective equations. Substituting(4.21) into (4.14) we have

(4.32)

$$
\begin{aligned}
&\frac{d^2}{dt^2}\int_{\mathbf{U}}\mathbf{u}_0\boldsymbol{\phi}+\int_{\mathbf{U}}\int_{\mathbf{Y}_s}Ae_x(\mathbf{u}_0):e_x(\boldsymbol{\phi})\\
&+\int_{\mathbf{U}}\int_{\mathbf{Y}_s}A\sum_{ij}\Big\{e_y(\mathbf{M}^{ij})(y)e_x(\mathbf{u}_0)_{ij}(t,x)+\int_0^t e_y(\mathbf{K}^{ij})(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)ds\Big\}:e_x(\boldsymbol{\phi})\\
&+\int_{\mathbf{U}}\int_{\mathbf{Y}_s}B[e_x(\partial_t\mathbf{u}_0)]:e_x(\boldsymbol{\phi})+\int_{\mathbf{U}}\int_{\mathbf{Y}_s}B\sum_{ij}\Big\{e_y(\mathbf{M}^{ij})(y)e_x(\partial_t\mathbf{u}_0)_{ij}(t,x)\\
&+e_y(\mathbf{K}^{ij})(0)(e_x(\mathbf{u}_0))_{ij}+\int_0^t e_y(\partial_t\mathbf{K}^{ij})(y,t-s)(e_x(\mathbf{u}_0))_{ij}(x,s)ds\Big\}:e_x(\boldsymbol{\phi})\\
&+\int_{\mathbf{U}}\int_{\mathbf{\Gamma}}\partial_t\mathbf{u}_0\cdot\boldsymbol{\phi}-\int_{\mathbf{U}}\int_{\mathbf{\Gamma}}(\partial_t\mathbf{u}_0\cdot\mathbf{n})(\boldsymbol{\phi}\cdot\mathbf{n})+\int_{\mathbf{U}}\int_{\mathbf{\Gamma}}h(\mathbf{u}_0\cdot\mathbf{n})(\boldsymbol{\phi}\cdot\mathbf{n})+\int_{\mathbf{U}}\int_{\mathbf{\Gamma}}g\boldsymbol{\phi}\cdot\mathbf{n}\\
&=\int_{\mathbf{U}}\mathbf{f}\cdot\boldsymbol{\phi}.
\end{aligned}
$$

Next, define the symmetric tensors $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ by

(4.33)

$$
\mathcal{A}_{ijkl}=\int_{\mathbf{Y}_s}A_{ijkl}+\int_{\mathbf{Y}_s}\sum_{mn}\frac{1}{2}A_{klmn}\left(\frac{\partial M^{ij}_n}{\partial y_m}+\frac{\partial M^{ij}_m}{\partial y_n}\right)+\frac{1}{2}B_{klmn}\left(\frac{\partial K^{ij}_n(0)}{\partial y_m}+\frac{\partial K^{ij}_m(0)}{\partial y_n}\right),
$$

$$
\mathcal{B}_{ijkl}=\int_{\mathbf{Y}_s}B_{ijkl}+\int_{\mathbf{Y}_s}\sum_{mn}\frac{1}{2}B_{klmn}\left(\frac{\partial M^{ij}_n}{\partial y_m}+\frac{\partial M^{ij}_m}{\partial y_n}\right),
\tag{4.34}
$$

$$\mathcal{C}_{ijkl}(t) = \int_{\mathbf{Y}_s} \sum_{mn} \left\{ \frac{1}{2} A_{klmn} \left(\frac{\partial K_n^{ij}}{\partial y_m} + \frac{\partial K_m^{ij}}{\partial y_n} \right) + \frac{1}{2} B_{klmn} \left(\frac{\partial(\partial_t K_n^{ij})}{\partial y_m} + \frac{\partial(\partial_t K_m^{ij})}{\partial y_n} \right) \right\}, \tag{4.35}$$

and let

$$N_{ij} = |\Gamma|\delta_{ij} - \int_{\mathbf{\Gamma}} (\mathbf{n} \otimes \mathbf{n})_{ij}, \; H_{ij} = \int_{\mathbf{\Gamma}} h(\mathbf{n} \otimes \mathbf{n})_{ij}, \; \mathbf{f}_1 = \int_{\mathbf{\Gamma}} g\mathbf{n}. \tag{4.36}$$

From (4.32), we obtain the effective equation for $\mathbf{u}_0$:

$$\partial_{tt}\mathbf{u}_0 - \text{div}\; [\mathcal{A}e(\mathbf{u}_0) + \mathcal{B}e(\partial_t \mathbf{u}_0)] - \text{ div } \int_0^t \mathcal{C}(t-s)e(\mathbf{u}_0)ds + N \cdot \partial_t \mathbf{u}_0 + H \cdot \mathbf{u}_0 = \mathbf{f} - \mathbf{f}_1. \tag{4.37}$$

The above calculations prove

Theorem 4.8. *Let $\mathbf{u}_0(t,x)$ be as in Lemma 4.5, then $\mathbf{u}_0(t,x)$ satisfies equation (4.37) and (4.16).*

From the effective equation (4.37), we find the effective stress tensor

$$\boldsymbol{\sigma}_0 = \mathcal{A}e(\mathbf{u}_0) + \mathcal{B}e(\partial_t \mathbf{u}_0)] - \int_0^t \mathcal{C}(t-s)e(\mathbf{u}_0)ds \tag{4.38}$$

and the effective drag force

$$\mathbf{D}_0 = N \cdot \partial_t \mathbf{u}_0 + H \cdot \mathbf{u}_0. \tag{4.39}$$

5. Well-posedness for cell problems and effective problem

Lemma 5.1. *The problem (4.25) is uniquely solvable.*

Proof. Since B is positive definite, the lemma follows from Korn's inequality and the Lax-Milgram lemma. □

Lemma 5.2. *The problem (4.28) is uniquely solvable.*

Proof. Since B is positive definite, the lemma follows from Korn's inequality and the Lax-Milgram lemma. □

Lemma 5.3. *The problem (4.31) is uniquely solvable.*

Proof. Let $\mathcal{K}^{ij}(\gamma) = \mathcal{L}\mathbf{K}^{ij}$, where $\mathcal{L}$ is Laplace transform. For properties of Laplace transform of distributions, we refer to [9]. Then $\mathcal{K}^{ij}$ satisfies

$$\operatorname{div}_y\Big\{(B\gamma + A)e_y(\mathcal{K}^{ij}) - Be_y(K_0^{ij})\Big\} = 0. \tag{5.1}$$

Since $B\gamma + A$ is positive definite for $\gamma > 0$, Korn's inequality and the Lax-Milgram lemma imply that (5.1) is uniquely solvable, so the lemma follows. □

Lemma 5.4. *The tensor $\mathcal{B}$ defined in (4.34) is positive definite.*

Proof. Let Λ be any symmetric matrix, then

$$\Lambda = \sum_{ij} \lambda_{ij} \frac{e_i \otimes e_j + e_j \otimes e_i}{2}.$$

Let

$$\mathbf{M}^\lambda = \sum_{ij} \lambda_{ij} \mathbf{M}^{ij}.$$

Then

$$\begin{aligned}
\mathcal{B}\Lambda : \Lambda &= \int_{\mathbf{Y}_s} B(\Lambda + e_y(\mathbf{M}^\lambda)) : \Lambda \\
&= \int_{\mathbf{Y}_s} B(\Lambda + e_y(\mathbf{M}^\lambda)) : (\Lambda + e_y(\mathbf{M}^\lambda)) - \int_{\mathbf{Y}_s} B(\lambda + e_y(\mathbf{M}^\lambda)) : e_y(\mathbf{M}^\lambda) \\
&= \int_{\mathbf{Y}_s} B(\Lambda + e_y(\mathbf{M}^\lambda)) : (\Lambda + e_y(\mathbf{M}^\lambda)) \\
&\geq \lambda_2 |\Lambda + e_y(\mathbf{M}^\lambda)|^2,
\end{aligned}$$

so the lemma follows. □

Lemma 5.5. *Let $\hat{\mathcal{C}}(\gamma)$ be the Laplace transform of $\mathcal{C}(t)$, then $\mathcal{A} + \hat{\mathcal{C}}(\gamma)$ is positive definite for $\gamma > 0$.*

Proof. Let

$$\hat{\mathbf{K}}^\lambda = \sum_{ij} \lambda_{ij} \hat{\mathbf{K}}^{ij},$$

where $\hat{\mathbf{K}}^{ij}$ is the Laplace transform of $\mathbf{K}^{ij}$.

Then

$$
\begin{aligned}
&(\mathcal{A}+\hat{\mathcal{C}})\Lambda:\Lambda\\
&=\int_{\mathbf{Y}_s}[A(\Lambda+e_y(\mathbf{M}^\lambda)]:\Lambda+\int_{\mathbf{Y}_s}[Ae_y(\hat{\mathbf{K}}^\lambda)+\gamma Be_y(\hat{\mathbf{K}}^\lambda)]:\lambda\\
&=\int_{\mathbf{Y}_s}[A(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))]:(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))+\int_{\mathbf{Y}_s}[\gamma Be_y(\hat{\mathbf{K}}^\lambda)]:\Lambda\\
&-\int_{\mathbf{Y}_s}[A(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))]:(e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))\\
&=\int_{\mathbf{Y}_s}[A(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))]:(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))\\
&+\int_{\mathbf{Y}_s}[\gamma Be_y(\hat{\mathbf{K}}^\lambda)]:(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))\\
&=\int_{\mathbf{Y}_s}[A(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))]:(\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda))+\int_{\mathbf{Y}_s}[\gamma Be_y(\hat{\mathbf{K}}^\lambda)]:e_y(\hat{\mathbf{K}}^\lambda)\\
&\geq\lambda_1|\Lambda+e_y(\mathbf{M}^\lambda)+e_y(\hat{\mathbf{K}}^\lambda)|^2+\gamma\lambda_2|e_y(\hat{\mathbf{K}}^\lambda)|^2,
\end{aligned}
$$

so the lemma follows. □

Lemma 5.6. *The homogenized equation (4.37) with initial condition* $\mathbf{u}_0(0)=u_0'(0)=0$ *has a unique solution.*

Proof. Taking Laplace transform in (4.37), we have

$$
\text{div }[\mathcal{A}+\gamma\mathcal{B}+\hat{\mathcal{C}}(\gamma)]\hat{u}_0-\gamma^2\hat{u}_0-\gamma(N-H)\cdot\hat{u}_0=-(\mathbf{f}-\mathbf{f}_1). \tag{5.2}
$$

By the above lemmas, $\mathcal{A}+\gamma\mathcal{B}+\hat{\mathcal{C}}(\gamma)$ is positive definite. Thus, Korn's inequality and the Lax-Milgram lemma imply that (5.2) is uniquely solvable, so the lemma follows. □

References

[1] G. Allaire, *Homogenization of the Navier-Stokes equations with a slip boundary condition,* Comm. pure and appl. math. **44** (1991) 605-641.

[2] G. Allaire, *Homogenization and two-scale convergence,* SIAM J. Math. Anal. **23.6** (1992) 1482-1518

[3] G. Allaire, A. Damlamian, U. Hornung, *Two-scale convergence on periodic surfaces and applications,* Mathematical modelling of flow thyough porous media, editors A. P. Bourgeat, C. Carasso, S. Luckhaus, A. Mikelic, (1995) 15-25.

[4] M. J. Buckingham, *Theory of acoustic attenuation, dispersion, and pulse propagation in unconsolidated granular materials, including marine sediments*, J. Acoust. Soc. Am. **102**,(1997), 2579-2596.

[5] M. J. Buckingham, *Theory of compressional and shear waves in fluid-like marine sediments*, J. Acoust. Soc. Am. **103**,(1998), 288-299.

[6] M. J. Buckingham, *Wave propagation, stress relaxation, and grain-to-grain shearing in saturated, unconsolidated marine sediments*, J. Acoust. Soc. Am. **108** (6), (2000), 2796-2815.

[7] R. Burridge and J. B. Keller, *Poroelasticity equations derived from microstructure*, J. Acoust. Soc. Amer., **70** (1981), 1140-1146.

[8] Clopeau, Th., Ferrín, J.L., Gilbert, R. P. and A. Mikelic, *Homogenizing the acoustic properties of the seabed, Part II*c), Modeling in Mathematics and Computation, 33(2001) 821-841.

[9] R. Dautray, J.L. Lions, *Mathematical analysis and Numerical methods for science and technology* vol.5, Evolution problems 1, Springer, Berlin, 1992.

[10] G. Duvaut, J.L. Lions, *Inequalities in mechanics and physics,* Springer-Verlag, 1972.

[11] C. Eck, J. Jarusek, Existence results for the semicoercive static contact problem with Coulomb friction Nonlinear analysis 42 (2000) 961-976.

[12] R. P. Gilbert and A. Panchenko, *Acoustics of a stratified poroelastic composite*, Zeitschrift für Analysis und ihre Anwendungen, **18**(4) (1999) 977-1001.

[13] R. P. Gilbert and A. Panchenko, *Effective acoustic equations for a two-phase medium with microstructure.* In press, Mathematical and computer modelling.

[14] R. P. Gilbert, A. N. Panchenko and X. Xie, *A prototype homogenization model for acoustics of granular materials* submitted, 2003.

[15] R. P. Gilbert and A. Mikelic, *Homogenizing the acoustic properties of the seabed: Part I*, Nonlinear Analysis, **40** (2000) 185-212.

[16] V. M. Harik, R. P. Gilbert and A. N. Panchenko, *Vibration of of bonded periodic composites: effects of the interface and distinct periodic structures.* International Journal of Solids and Structures, 40 (2003) 3177-3193.

[17] K. Kuttler, M. Shillor, *Set-valued pseudomonotone maps and degenerate evolution inclusions,* Comm. Contemp. math.**1**(1) (1999) 87-123.

[18] J.A.C. Martins, J.T. Oden, *Existence and uniqueness results for dynamic contact problems with non-linear normal and friction interface laws,* Nonlinear anal. vol 11, (1987) 407-428.

[19] M. Neuss-Radu, *Some extensions of two-scale convergence,* C.R. Acad.Sci. Paris **t. 322 Serie I** (1996) 899-904.

[20] G, Nguetseng, *A general convergence result for a functional related to the theory of homogenization,* SIAM J. Math. Anal. **20** (1989) 608-623.

[21] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theopry* Lecture Notes in Physics **127** Springer, Berlin (1980)

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