

# Effective Acoustic Equations for a Two-phase Medium with Microstructure.

Robert P. Gilbert\*

Department of Mathematical Sciences

University of Delaware

and

Alexander Panchenko†

Department of Mathematics

Washington State University

March 25, 2004

## Abstract

We study acoustic wave propagation in a two-phase medium in which the solid phase is a linear elastic material, and the fluid phase is assumed to be a compressible Newtonian barotropic fluid. Assuming that properties of the medium change rapidly on the small scale  $\epsilon$ , we analyze the microscopic nonlinear Navier-Stokes equations and show that they can be linearized when  $\epsilon$  tends to zero. Using a variant of Tartar's method of oscillating test functions, we derive effective acoustic equations which turn out to be viscoelastic. In order to treat disordered materials occurring in nature, we develop a new approach to describing geometry of a non-periodic medium with length scale separation. Our approach is not based on probabilistic considerations. Instead, we postulate that certain inequalities hold uniformly on the micro-scale.

## 1 Introduction

Averaging solid-fluid composites with instantaneous memory typically produces history dependent viscoelastic effective equations ([13, 5, 12, 7, 6]), while the specific mechanism of the microscopic energy dissipation is reflected in the choice of the constitutive functions. Motivated by this, we have attempted to lay a fairly general theoretical foundation for derivation of the viscoelastic models for acoustics of two-phase materials with microstructure.

---

\*This work was supported in part by ONR grant N00014-001-0853

†This work was supported in part by ONR grant N00014-001-0853

The size of a typical microstructural element (e.g. a pore cavity) is characterized by a so-called microscopic length scale which is much smaller than the sample size. The microscopic length scale  $\epsilon$  is introduced by assuming that material properties are of the form  $f(x, \frac{x}{\epsilon})$ , where  $y = \frac{x}{\epsilon}$  is the so-called fast variable. Using techniques of homogenization, the actual highly inhomogeneous material can be replaced by a quasi-uniform effective medium with the material properties of the form  $f^{eff}(x)$ , variations of which are significant only over distances comparable to the size of the whole sample.

Previously work in this field [13, 5, 12, 6, 7] was limited to analysis of linear microscopic equations. In addition, the authors of [5, 12, 7] assumed that the propagating wave is time-harmonic, which reduces the evolutionary problem to a quasi-static one. In this paper we treat the time dependent case assuming that the fluid phase is described by the nonlinear equations of a compressible viscous barotropic fluid. We show that if the fluid is only slightly compressible and microstructure is present, then the velocity  $\mathbf{v}$  of the fluid can be written as  $\mathbf{v} = \mathbf{w} + \bar{\mathbf{v}}$  where the relative velocity  $\mathbf{w}$  tends to zero in a suitable sense as  $\epsilon$  goes to zero, and  $\bar{\mathbf{v}}$  satisfies linearized equations which we subsequently average to obtain linear viscoelastic effective equations.

Another important question we are concerned with is geometric assumptions. This question can not be avoided if one wishes to develop rigorous methods for derivation of averaged equations. Earlier work can be classified into two groups, namely semi-empirical works where details of geometry on the microscale are ignored [2], [3] [4]. More rigorous works [13, 12, 7, 6] employed a periodic geometric structure, with material properties of the form  $f(\frac{x}{\epsilon})$ , or slowly-varying quasi-periodic structure, for which material properties are given by functions  $f(x, \frac{x}{\epsilon})$ , periodic in the fast variable. However, in non man-made materials the periodicity assumption is seldom realized. To deal with geometric irregularities one could try to use the probabilistic formalism introduced by Kozlov and related notions of random porous domains ([1]). Unfortunately, this formalism does not preclude existence of arbitrary large balls containing only fluid. This would mean that the material contains arbitrary large pores which is quite different from the intuitive picture of "homogenization-suitable" materials. Even more important is the fact that many natural materials like porous rocks and marine sediments exhibit only relatively small inhomogeneities. Since this is precisely the situation we wish to deal with, a different geometric framework is needed. To develop such a framework, we sought a model in which the length scale separation is maintained but the overly restrictive periodicity assumption is dropped. Our assumptions might be classified as ones preserving bulk geometric properties that occur uniformly on the microscale, but still allow geometric details to vary from point to point. Roughly speaking, we use the following geometric postulates.

- It is possible to cover the region we are concerned with by disjoint cubes  $Q_k^\epsilon$  of side length proportional to a small parameter  $\epsilon$  and the volume fraction of the fluid phase is approximately the same for each cube.
- The typical size of inhomogeneities cannot be much smaller than  $\epsilon$ .

If the spatial distribution of solid and fluid satisfies the two assumptions above, we call the corresponding geometry *an admissible two-scale geometry*. An example of admissible geometry is shown on Fig. 1.

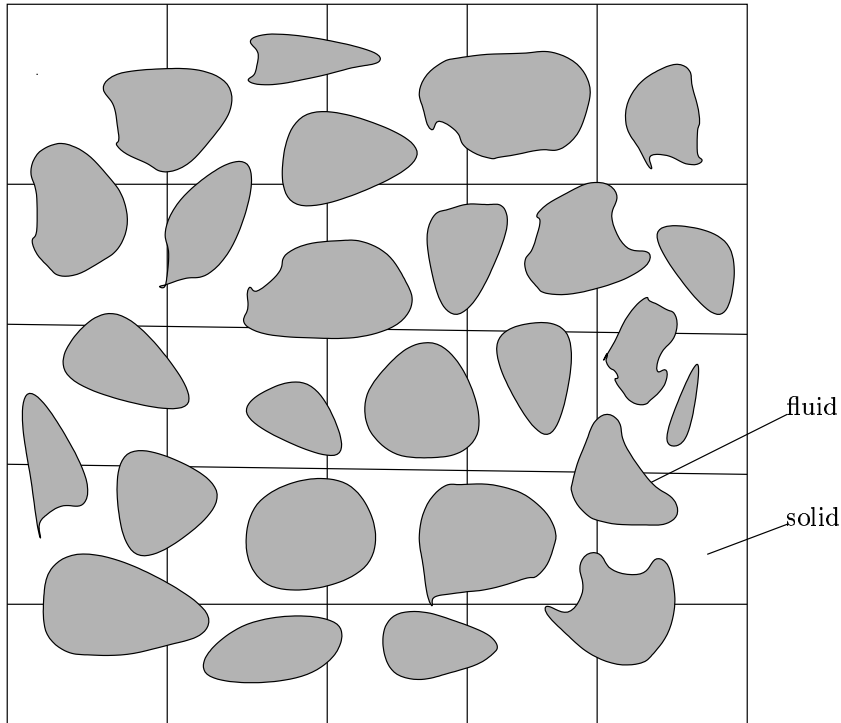


Figure 1: Composite medium with scale separation.

Next two assumptions are related to the a priori smallness of the acoustic displacements.

- The fluid phase has small compressibility.
- The time-dependence of the characteristic function for the fluid domain does not influence the "effective equations".

These four assumptions will be made precise in the course of our discussions. We also provide additional comments on their implications.

The paper is organized as follows. The microscopic model of a two-phase medium is described in Section 2. In Section 3 we list geometric assumptions on the medium and note some implications. Small compressibility assumption is analyzed in Section 4. In Section 5, the linearized microscopic equations are obtained. Section 6 contains the derivation of the effective equations. Finally, in Section 7 we comment on some mathematical issues involved.

## 2 Microscopic Equations.

We consider a composite medium that contains a porous elastic phase and a fluid phase. Thus our model can be considered solid-like. We point out, however, that a crude but feasible continuum model of the granular flow obtained from the kinetic theory considerations is that of a compressible viscous fluid [9]. So our model may be also viewed as the approximation of the partially consolidated material containing a granular phase.

We postulate the following microscopic equations.

*Solid Phase.*

By  $\mathbf{u}$  we denote the displacement in the solid.

$$\rho_0(x)\partial_t^2\mathbf{u} - \operatorname{div}(A(x)e(\mathbf{u})) = f, \quad (2.1)$$

where  $\rho_0$  is a known density,  $A$  is an elastic tensor,  $e(\mathbf{u})$  denotes symmetric part of the deformation gradient, and  $f$  is density of a body force. We assume that  $A$  is symmetric and its components are bounded from above and below.

*Fluid Phase.*

The state variables here are density  $\rho$ , velocity  $\mathbf{v}$  and pressure  $P$ . We use the system of equations of compressible barotropic fluid.

Mass balance:

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0. \quad (2.2)$$

Momentum balance:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v} \otimes \mathbf{v}) - \mu\Delta\mathbf{v} - \xi\operatorname{div}\mathbf{v} + \nabla P = f, \quad (2.3)$$

where  $(\mathbf{v} \otimes \mathbf{v})_{ij} = v_i v_j$ .

Equation of state:

$$P = a\rho^\delta, \quad (2.4)$$

where  $a$  is a material constant and  $\delta$  is larger than 1. In (2.3)  $\mu$  and  $\xi$  are viscosity coefficients.

The initial conditions are

$$\mathbf{v}(0, x) = \mathbf{v}^0(x), \quad \partial_t\mathbf{v}(x, 0) = 0. \quad (2.5)$$

On the exterior boundary of the domain containing the sample, zero Dirichlet condition for  $\mathbf{v}$  is prescribed.

Finally, we comment on the conditions on the interface between the phases. In the actual motion, the fluid and solid domains change in time so that each of these domain are stationary in the referential (but not spatial) description. On the actual evolving interface, one should prescribe continuity of velocity and traction. In this paper, we use spatial description together with the quasi-static interface assumption (see Assumption 3.1 below). If continuity of traction is imposed on the static interface, the weak form of the momentum balance equation

would contain non-physical terms supported on the interface. To avoid this, we simply require that the interface conditions on the solid and fluid tractions should be such that no interface-supported terms appear in the weak formulation of the momentum equation.

### 3 Geometry of the Problem.

Here we list all the assumptions concerning the geometry of phases. Consider a fixed large cube  $U$  in the three-dimensional space. Since we are using effective equations for the granular phase, we can assume that there are no voids. Then  $U$  can be written as a union of two non-intersecting domains: the solid domain  $W^\epsilon$  and the fluid domain  $V^\epsilon$ . The subscript indicates dependence on the small parameter  $\epsilon$ . For a fixed cube  $C$  define  $C^\epsilon$  to be the cube shrunk by a factor of  $\epsilon$ :  $C^\epsilon = \{x : \epsilon^{-1}x \in C\}$ .

**Assumption 3.1** *For any  $0 < \epsilon \leq 1$ ,  $U$  can be broken up into disjoint cubes  $C_k^\epsilon$  with the properties*

$$C_1\epsilon^3 \leq \text{vol}(C_k^\epsilon) \leq C_2\epsilon^3 \quad (3.1)$$

with  $C_1, C_2$  independent of  $\epsilon, k$ .

The volume fraction  $\alpha_k^\epsilon$  of the solid phase in each cube satisfies

$$C_3 \leq \alpha_k^\epsilon \leq C_4, \quad (3.2)$$

with  $C_3, C_4$  independent of  $\epsilon, k$  and such that  $0 < C_3 < C_4 < 1$ .

Next we consider the characteristic function  $\theta(x, t, \epsilon)$  of the fluid domain  $V^\epsilon$ . In fact, to avoid mathematical technicalities in the sequel, we will take  $\theta$  to be a smoothed-out characteristic function. From now on we will assume that  $0 \leq \theta \leq 1$  is smooth, and differs from zero in the domain which is only slightly larger than  $V^\epsilon$ .

It is clear that Assumption 3.1 forces spatial oscillations of  $\theta$  with the frequency comparable to  $\epsilon$ .

An important mathematical consequence of this assumption is that for every  $C^1$ -function  $f$  on  $C_k$  such that  $f = 0$  on  $W^1 \cap C_k$ , the Poincaré inequality

$$\int_{C_k} |f|^2(x) dx \leq L \int_{C_k} |\nabla f|^2(x) dx \quad (3.3)$$

holds with a constant  $L$  independent of the choice of  $C_k$ .

It is shown in ([16], Sect. 4.1 th 4.4.2 and Corollary 4.4.3) that the constant in the Poincaré inequality depends only on the cube  $C_k$ , and the Lebesgue measure of the set on which  $f = 0$ . Using (3.1, 3.2) we can find a constant  $L$  which is the same for all cubes.

In the context of acoustics one typically assumes that displacements are small. This means that  $V^\epsilon$  is close to its initial shape at all times. In this paper we are going to impose a more stringent assumption and assume that  $V^\epsilon$  is fixed.

**Assumption 3.2** *Time-dependence of  $\theta$  does not influence the effective equations and thus can be ignored.*

This assumption is implicitly present in virtually all papers cited above. It is necessary for the validity of the quasi-static (time-harmonic) equations in [5], [12]. If one works with linear evolution equations and uses Laplace transform in time as in [13], [6], time-independence of the coefficients is also crucial. On the other hand, the assumption that  $\theta$  is time independent seems to be important if one wishes to avoid analysis of the contact conditions between parts of the solid phase. For instance, if most of the fluid phase is contained in narrow channels, one could specify the initial conditions so that vibrations of the walls of the channels would produce rubbing of the walls against each other.

We wish to study a medium in which a micro-length scale of order  $\epsilon$  is well separated from a macro-scale of order 1. To make this notion more precise, we impose an additional assumption.

**Assumption 3.3** *There exists a function  $Q(x, y)$ , where  $x$  ranges over  $U$ , and  $y$  belongs to  $\mathbf{R}^3$ , with the following properties.*

$$\theta(x, \epsilon) = Q\left(x, \frac{x}{\epsilon}\right), \quad (3.4)$$

for  $0 < \epsilon \leq 1$ , and partial derivatives of  $Q$  are bounded uniformly.

This assumption explicitly introduces the separation of scales by separating dependence on the fast variable  $y = \frac{x}{\epsilon}$  from the dependence on the slow variable  $x$ .

While being a common feature of numerous papers and books on homogenization assumptions of this sort are implicit, because a real medium is modeled by functions  $\theta(x, \epsilon)$ , and nothing about  $Q$  is known a priori. So the question behind Assumption 3.3 is whether  $Q$  can be constructed given a family of functions  $\theta(x, \epsilon)$  with observable properties.

Assuming that  $Q$  is found and  $\theta$  is a continuously differentiable function of  $x, \epsilon$  for  $0 < \epsilon \leq 1$ , we see that necessarily

$$|\nabla_x \theta(x, \epsilon)| = \left| \nabla_x Q\left(x, \frac{x}{\epsilon}\right) + \frac{1}{\epsilon} \nabla_y Q\left(x, \frac{x}{\epsilon}\right) \right| \leq \left( C_1 + \frac{C_2}{\epsilon} \right). \quad (3.5)$$

and

$$|\partial_\epsilon \theta(x, \epsilon)| = \left| -\frac{1}{\epsilon^2} x \cdot \nabla_y Q\left(x, \frac{x}{\epsilon}\right) \right| \leq \frac{C_2}{\epsilon^2}. \quad (3.6)$$

This shows that Assumption 3.3 places restrictions on the frequency of oscillations of the function  $\theta$ . We therefore interpret it by saying that the local

geometric inhomogeneities can not be much smaller than  $\epsilon$ . Note that while Assumption 3.3 curbs oscillations of  $\theta$  from above, Assumption 3.1 restricts them from below.

**Remark.** Of course, it would be nice to have some conditions on  $\theta$  that imply existence of  $Q$ . In particular, one might ask if it is possible to construct  $Q$  starting with  $\theta$  satisfying (3.5) and (3.6). A partial answer to this question is given in Section 7.

We emphasize that the above restrictions are flexible and geometry on the microscale is not rigidly specified. In particular we do not assume periodicity or almost periodicity in the fast variable.

This fact makes it useful to compare our model with the probabilistic description of random domains introduced by Kozlov (see, for instance, [1]). Let  $\omega$  be an element of the probability space  $\Omega$  equipped with a probability measure  $\nu$ . Assuming that  $\theta$  is a stationary random field and making a suitable choice of  $\Omega, \nu$  one can find a  $\nu$ -preserving family of invertible transformations  $T(x)$  on  $\Omega$  such that  $\theta(x, 1, \omega) = \theta(T(x)\omega)$ . In addition,  $T(x)$  satisfies  $T(0) = I$ ,  $T(x + y) = T(x)T(y)$  (see [10]) for details. Next, one defines

$$\theta(x, \epsilon, \omega) = \theta\left(T\left(\frac{x}{\epsilon}\right)\omega\right).$$

According to [1],  $V^\epsilon$  is a random domain if

$$V^\epsilon = \{y \in \mathbf{R}^3 : T(y)\omega \in V^0\},$$

where  $V^0$  is a fixed subset of the probability space  $\Omega$ .

In addition, one calls a random domain *porous* if there exists a random variable  $h(\omega) > 0$  such that  $\int h^{-1}d\nu < \infty$  and for any smooth compactly supported in  $V^\epsilon$  function  $\phi$ ,

$$\int h(T(y)\omega)\phi^2(y)dy \leq \int |\nabla\phi(y)|^2dy.$$

This inequality can be seen as a Poincaré with some mild degeneration allowed, since the weight function  $h$  can be small on large subsets of  $V^\epsilon$ .

If we choose  $\Omega$  to be torus of unit radii and  $T(x)\omega = x + \omega(\text{mod } \mathbf{Z}^3)$ , we obtain a periodic domain  $V^\epsilon$  of period  $\epsilon$ . In this case it is easy to see that the Poincaré inequality (3.3) holds with  $C_k$  being identical cubes of side length one [15]. The periodic domains are also porous in the sense of [1], since the function  $h$  can chosen to be a constant.

However, Assumption 3.1 and related inequality (3.3) will not hold for an arbitrary random geometry (see [1]), since probabilistic description allows for existence of arbitrary large cubes that lie entirely in  $V^\epsilon$ . This runs contrary to our intuitive picture of a medium with a large number of small inhomogeneities. Another aspect of the probabilistic approach is that homogenized equations are constant-coefficient, which can be viewed as a consequence of the fact that the random field  $\theta$  is stationary.

In this paper we aim to avoid references to the probability theory and the theory of dynamical systems. Instead, we try to impose some natural conditions (on volume fractions, oscillations of material coefficients etc.) that can be verified in experiments. Our approach to describing geometric complexity of the model is restrictive in terms of admissible length scales but otherwise rather flexible. The main ingredient is to postulate that certain inequalities (e.g. Poincaré) hold *uniformly* on the small length scale(s). We believe that this approach can be adapted to a variety of situations common in homogenization of non man-made materials.

## 4 Small Compressibility and Its Implications.

In what follows we will need to compare spatial (Eulerian) and referential (Lagrangian) descriptions of flows. Let  $X$  denote an initial position of a fluid particle. It is used as a "label" of a moving particle in the referential description. The position  $x$  at a time  $t$  is related to  $X$  via the deformation map  $\chi$  defined by  $x = \chi(X, t)$ . We assume that  $\chi$  is invertible for almost all  $t$  and satisfies an ordinary differential equation  $\partial_t \chi(X, t) = \mathbf{v}(\chi(X, t), t)$  for almost all  $X$ . The displacement is defined as  $\mathbf{u}(X, t) = x - X = \chi(X, t) - X$ .

Consider the conservation of mass equation

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (4.1)$$

with the initial condition  $\rho(x, 0) = \rho_0$ , where  $\rho_0$  is a constant.

Our goal is to investigate the following formal linearization procedure. First, one writes  $\rho = \rho_0 + \rho_1$ , where  $\rho_1$  is supposed to be small. Substituting this into (4.1) and dropping terms containing spatial derivatives of  $\rho_1$ , one obtains  $\rho_1 = -\rho_0 \int_0^t \operatorname{div} \mathbf{v}(x, \tau) d\tau$ , so the excess (linearized) pressure  $P = a\rho_1$  is approximated by  $-a\rho_0 \operatorname{div} \int_0^t \mathbf{v}(x, \tau) d\tau$ . In particular, this type of reduction has been used implicitly in papers of Burridge and Keller [5] and Buckingham [4], and also in [7]. So, it is of obvious interest to see what conditions one has to impose for the validity of the above procedure.

We introduce the functions  $\tilde{\rho}(X, t) = \rho(\chi(X, t), t)$ ,  $\tilde{\rho}_1(X, t) = \rho_1(\chi(X, t), t)$ . Then (4.1) reads

$$\partial_t \tilde{\rho} + F(X, t) \tilde{\rho} = 0, \quad (4.2)$$

where  $F(X, t) = \operatorname{div} \mathbf{v}(\chi(X, t), t)$ . (Note that divergence is still with respect to the spatial variable  $x$ ). Then, integrating, we obtain

$$\tilde{\rho} e^{\int_0^t F(X, \tau) d\tau} = C(X), \quad (4.3)$$

and from the initial condition  $C(X) = \rho_0$ . Then

$$\tilde{\rho}_1(X, t) = \rho_0 (e^{-\int_0^t F(X, \tau) d\tau} - 1).$$

In the homogenization problem, one has to deal with a family of density functions depending on the microstructural parameter  $\epsilon$ . The corresponding



family of the functions  $F$  also depends on  $\epsilon$ . On this family we impose the *Small Compressibility Assumption*.

**Assumption 4.1**

$$\left| \int_0^t F(X, \tau, \epsilon) d\tau \right| \leq \gamma$$

where  $\gamma$  is independent of  $t$  and can be chosen the same for all  $\epsilon \in (0, 1]$ .

This assumption can be conveniently formulated in terms of the Jacobian determinant  $J$  of the deformation gradient defined by  $J(X, t) = \det \nabla \chi(X, t)$ . Using the Euler's identity

$$\frac{\partial_t J}{J}(X, t) = \operatorname{div} \mathbf{v}(\chi(X, t), t)$$

we get

$$\int_0^t F(X, \tau) d\tau = \log J(X, t).$$

Then small compressibility assumption requires  $\log J$  to be less than  $\gamma$  at all times. In the limit  $\gamma \rightarrow 0$  we recover the incompressibility constraint  $J = 1$ .

Another useful way of interpreting the small compressibility assumption is to relate it to the bounds on density. Writing the conservation of mass equation in the referential description

$$J(X, t) \tilde{\rho}(X, t) = \rho_0 J(x, 0) = \rho_0,$$

we obtain  $\log J = \log \rho_0 - \log \tilde{\rho}$  from which we see that the Assumption 5.1 is equivalent to the inequality

$$\log \rho_0 - \gamma \leq \log \tilde{\rho} \leq \log \rho_0 + \gamma, \quad (4.4)$$

which means that the density  $\tilde{\rho}$  (and hence  $\rho$ ) is controlled independent of  $X, t$ , and small compressibility parameter  $\gamma$  determines the maximal possible magnitude of the density perturbation with respect to the initial density  $\rho_0$ . Since the bounds are position-independent, we can drop tilde in the inequality (4.4) and exponentiate to obtain

$$\rho_0 e^{-\gamma} \leq \rho(x, t) \leq \rho_0 e^{\gamma}. \quad (4.5)$$

In the situation involving homogenization, (4.5) is supposed to hold uniformly in  $\epsilon$ .

Next, we define  $G(X, t) = -\log J = -\int_0^t F(X, \tau) d\tau$  and write an asymptotic series:

$$\tilde{\rho}_1(X, t) \sim \rho_0 \sum_{j=0}^{\infty} G(X, t)^j, \quad (4.6)$$

as  $\gamma \rightarrow 0$ . Up to the terms of order  $\gamma^2$  we have a linearization

$$\tilde{\rho} = \rho_0 \left( 1 - \int_0^t \operatorname{div} \mathbf{v}(\chi(X, \tau), \tau) d\tau \right).$$

This yields an approximation for the linearized pressure  $P_0$ :

$$P_0(\chi(X, t), t) = a\rho_0 \left( 1 - \int_0^t \operatorname{div} \mathbf{v}(\chi(X, \tau), \tau) d\tau \right) + O(\gamma^2). \quad (4.7)$$

This equation is written in the referential description, so it is not directly suitable for use in the Eulerian fluid momentum equation. To replace (4.7) by the commonly used acoustic approximation

$$P(x, t) = a\rho_0 \left( 1 - \operatorname{div} \int_0^t \mathbf{v}(x, \tau) d\tau \right) + O(\gamma^2) = a\rho_0 \left( 1 - \operatorname{div} \mathbf{u}(x, t) \right) + O(\gamma^2), \quad (4.8)$$

we need to estimate the difference

$$\int_0^t \operatorname{div} \mathbf{v}(\chi(X, \tau), \tau) d\tau - \operatorname{div} \int_0^t \mathbf{v}(x, \tau) d\tau = \log J(X, t) - \log J(x, t).$$

Making use of the conservation of mass in the referential description we obtain

$$\int_0^t \operatorname{div} \mathbf{v}(\chi(X, \tau), \tau) d\tau - \operatorname{div} \int_0^t \mathbf{v}(x, \tau) d\tau = \log \tilde{\rho}(x, t) - \log \tilde{\rho}(X, t), \quad (4.9)$$

and hence

$$\begin{aligned} \left| \int_0^t \operatorname{div} \mathbf{v}(\chi(X, \tau), \tau) d\tau - \operatorname{div} \int_0^t \mathbf{v}(x, \tau) d\tau \right| &\leq & (4.10) \\ \left| \log \tilde{\rho}(x, t) - \log \rho_0 \right| + \left| \log \rho_0 - \log \tilde{\rho}(X, t) \right| &\leq 2\gamma. \end{aligned}$$

## 5 Linearization.

In this section we linearize the fluid equations (2.2)-(2.4). In the context of acoustics, inertial effects are assumed to be negligible. In the present case, however, this is not clear, since we do not assume that the small compressibility parameter tends to zero. Rather, we fix  $\gamma$  and then pass to the limit  $\epsilon \rightarrow 0$  in the governing equations. Therefore, we need to show that inertial terms are small with respect to the microstructure parameter, and not the small compressibility parameter. In order to do this, we write  $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{w}$  where  $\mathbf{w}$  is the relative velocity. We assume that  $\mathbf{w}(x, 0) = 0$  and  $\mathbf{w}(x, t) = 0$  on the interface between the phases. The function  $\bar{\mathbf{v}}$  is required to satisfy the linearized system of equations

$$\partial_t(P_0) + a\rho_0 \operatorname{div} \bar{\mathbf{v}} = 0, \quad (5.1)$$

$$\rho_0 \partial_t \bar{\mathbf{v}} - \mu \Delta \bar{\mathbf{v}} - \xi \operatorname{div} \nabla \bar{\mathbf{v}} + \nabla P_0 = f, \quad (5.2)$$

which can be rewritten as a single equation

$$\rho_0 \partial_t \bar{\mathbf{v}} - \mu \Delta \bar{\mathbf{v}} - \xi \operatorname{div} \bar{\mathbf{v}} + \nabla P_0 = f, \quad (5.3)$$

where  $P_0(x, t) = a\rho_0(1 - \int_0^t \operatorname{div} \bar{\mathbf{v}}(x, \tau) d\tau)$ . In Section 7, we show that there is  $C$  independent of  $t, \epsilon$  such that

$$\int_0^T \int_{V^\epsilon} |\mathbf{w}(x, t)|^2 dx = \int_0^T \int_U |\mathbf{w}(x, t)|^2 dx \leq C\epsilon^2. \quad (5.4)$$

This inequality implies that  $\mathbf{w}$  tends to zero as  $\epsilon$  goes to zero, so for small  $\epsilon$  most of the energy is carried by  $\bar{\mathbf{v}}$ . The effective behavior of the fluid phase is completely determined by the solution of the linearized equations for  $\bar{\mathbf{v}}$ . Physically, this means that solid and fluid vibrate in phase for small  $\epsilon$  and the relative motion of the fluid can be neglected.

The key role in the derivation of (5.4) is played by the Poincaré inequality (3.3) which in turn follows from the Assumption 3.1 on volume fractions.

## 6 Averaged Equations.

Let us define the displacement in  $V^\epsilon$  by

$$\mathbf{u} = \int_0^t \bar{\mathbf{v}}(x, \tau) d\tau.$$

Then the linearized equation for  $\bar{\mathbf{v}}$  is equivalent to

$$\rho_0 \partial_t^2 \mathbf{u} - \operatorname{div} (A^f e(\partial_t \mathbf{u}) + a\rho_0 \operatorname{div} \mathbf{u} I) = f, \quad (6.1)$$

where  $I$  is the second order identity tensor, and

$$A_{ijkl}^f = 2\mu \delta_{ik} \delta_{jl} + (\xi - \mu) \delta_{ijkl}.$$

In what follows we will use  $\theta(x, \frac{x}{\epsilon})$  to denote the characteristic function of  $V^\epsilon$ . Then the characteristic function of the solid domain  $W^\epsilon$  is  $1 - \theta$  in  $U$ . The elastic tensor of the solid will be denoted by  $A^s$ . Combining (6.1) with the elastic equations in the solid we obtain the system

$$\rho \partial_t^2 \mathbf{u} - \operatorname{div} T = f \quad (6.2)$$

in  $U$ , where the microscopic stress tensor  $T$  is given by

$$T = (1 - \theta) A^s e(\mathbf{u}) + \theta A^f e(\partial_t \mathbf{u}) + \theta a\rho_0 \operatorname{div} \mathbf{u} I. \quad (6.3)$$

In order to obtain the effective stress we multiply divergence of  $T$  by a certain parameter-dependent test function  $\phi^\epsilon$ , integrate by parts, and then pass to the limit in the resulting integrals. This procedure is known as the method of

oscillating test functions and is due to Tartar [14]. The crucial ingredient here is the choice of  $\phi^\epsilon$ . We set

$$\phi^\epsilon(x, t) = \phi(t, x) + \epsilon\phi_1(t, x, \frac{x}{\epsilon}),$$

where the function  $\phi$  is arbitrary, and  $\phi_1$  depends on  $\phi$ , namely,

$$\phi_1(t, x, y) = N^{pq}(x, y)e(\phi)_{pq} + \int_t^T M^{pq}(t - \tau, x, y)e(\phi)_{pq}(\tau, x) d\tau. \quad (6.4)$$

The the vectors  $N^{pq}$  and  $M^{pq}$  are found from the fast-variable problems. In the case of a periodic geometry, these problems reduce to the cell problems posed on the periodicity cell. To describe the fast-variable problems we need some notation. Denote by  $\varepsilon_{ij}$  the square matrix with the components  $\varepsilon_{ij}^{kl}$  equal to one if  $k = i, l = j$  and zero otherwise. Then  $N^{pq}$  is required to satisfy

$$\operatorname{div}_y [(1 - \theta)A^s + \theta A^f](e_y(N^{pq}) + \varepsilon_{pq}) = 0. \quad (6.5)$$

After  $N^{pq}$  are determined, we can find the initial value  $M_0^{pq}(y)$  by solving

$$\operatorname{div}_y (\theta[A^f(e(M_0^{pq}) + \varepsilon_{pq}) + a\rho_0 \operatorname{div}_y N^{pq}I]) = 0. \quad (6.6)$$

Finally, the  $M^{pq}$  are required to satisfy the equation

$$\operatorname{div}_y [(1 - \theta)A^s e(M^{pq}) - \theta A^f e(\partial_t M^{pq}) + \theta a\rho_0 \operatorname{div}_y M^{pq}I] = 0. \quad (6.7)$$

The problems (6.5)-(6.7) determine the vectors  $N^{pq}, M^{pq}$  in (6.4). Since these problems contain only differentiations with respect to  $t$  and  $y$ , it is natural to call them *fast-variable problems*.

To avoid componentwise notation in what follows, we define the fourth-order tensors  $K^1, K^2$  and the second-order tensors  $K^3, K^4$  as follows.

$$K_{ijkl}^1(y) = e_y(N^{ij})_{kl}, \quad (6.8)$$

$$K_{ijkl}^2(t, y) = e_y(M^{ij})_{kl},$$

$$K_{ij}^3 = \operatorname{div}_y N^{ij},$$

$$K_{ij}^4 = \operatorname{div}_y M^{ij}.$$

Next we observe that for a differentiable function  $\psi(x, y)$ ,

$$\partial_{x_j} \psi(x, \frac{x}{\epsilon}) = (\partial_{x_j} + \frac{1}{\epsilon} \partial_{y_j}) \psi(x, \frac{x}{\epsilon}).$$

Using this we obtain

$$- \int_0^T \int_U \operatorname{div} T \cdot \phi^\epsilon dx dt = \int_0^T \int_U T e(\phi^\epsilon) dx dt = \quad (6.9)$$

$$\begin{aligned} & \int_0^T \int_U [(1-\theta)A^s e(\mathbf{u}) + \theta a \rho_0 \operatorname{div} \mathbf{u} I] \cdot (e(\phi) + e_y(\phi_1)) - \theta A^f e(\mathbf{u}) \cdot \partial_t (e(\phi) + e_y(\phi_1)) \, dx \, dt \\ & + \epsilon \int_0^T \int_U (1-\theta)A^s e(\mathbf{u}) \cdot e_x(\phi_1) - \theta A^f e(\mathbf{u}) \cdot e_x(\partial_t \phi_1) + \theta a \rho_0 \operatorname{div} \mathbf{u} I \cdot e_x(\phi_1) \, dx \, dt. \end{aligned}$$

The last integral is bounded uniformly in  $\epsilon$  so that the corresponding term disappears in the limit  $\epsilon \rightarrow 0$ . Integrating by parts in the remaining integral we obtain

$$\int_0^T \int_U T \cdot e(\phi^\epsilon) \, dx \, dt = \frac{1}{\epsilon} \int_0^T \int_U \mathbf{u} \cdot \operatorname{div}_y (B_1 - B_2) \, dx \, dt + \quad (6.10)$$

$$\int_0^T \int_U \mathbf{u} \cdot \operatorname{div}_x (B_1 - B_2) \, dx \, dt + O(\epsilon),$$

where

$$B_1 = (1-\theta)A^s (e(\phi) + e_y(\phi_1)) + \theta a \rho_0 (\operatorname{div} \phi + \operatorname{div}_y \phi_1) I,$$

and

$$B_2 = \theta A^f (e(\partial_t \phi) + e_y(\partial_t \phi_1)).$$

Due to the choice of  $\phi_1$ , the  $\epsilon^{-1}$ -term in the above equation is zero. This can be verified by straightforward computation using the fact that  $N^{pq}$  and  $M^{pq}$  solve the local problems.

To obtain the effective stress, we need to pass to the limit in the  $\epsilon^0$ -term. This passage can be justified by estimating solutions of the fast-variable problems (6.5)-(6.7), as explained in Section 7.

Using the definition of  $\phi_1$  and the tensors  $K^j$  defined in (6.8) we write

$$B_1 - B_2 = T_1^\epsilon e(\phi) - T_2^\epsilon e(\partial_t \phi) + \int_0^t T_3^\epsilon(t - \tau, \cdot) e(\phi)(\tau, \cdot) \, d\tau,$$

where the tensors  $T_j^\epsilon$  are defined by

$$T_1^\epsilon = (1-\theta)A^s (I + K^1) - \theta A^f K^2(0, \cdot) + \theta K^3 I, \quad (6.11)$$

$$T_2^\epsilon = \theta A^f (I + K^1), \quad (6.12)$$

$$T_3^\epsilon = (1-\theta)A^s K^2 - \theta A^f \partial_t K^2 + K^4 I. \quad (6.13)$$

Now we find the effective tensors  $T_1(x), T_2(x), T_3(t, x)$  that satisfy

$$\operatorname{div} (T_1 e(\phi)) = \lim_{\epsilon \rightarrow 0} \operatorname{div}_x (T_1^\epsilon e(\phi)), \quad (6.14)$$

$$\operatorname{div} (T_2 e(\partial_t \phi)) = \lim_{\epsilon \rightarrow 0} \operatorname{div}_x (T_2^\epsilon e(\partial_t \phi)), \quad (6.15)$$

$$\operatorname{div} \int_t^T T_3(t - \tau, x) e(\phi)(\tau, x) d\tau = \lim_{\epsilon \rightarrow 0} \operatorname{div}_x \int_t^T (T_2^\epsilon(t - \tau, \cdot) e(\phi)(\tau, \cdot)) d\tau. \quad (6.16)$$

Returning to (6.9) we see that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^T \int_U T \cdot \phi^\epsilon dx dt &= \lim_{\epsilon \rightarrow 0} \int_0^T \int_U T \cdot \phi dx dt = \\ \int_0^T \int_U \mathbf{u}_0 \cdot \operatorname{div} [T_1 e(\phi) - T_2 e(\partial_t \phi) + \int_t^T T_3(t - \tau, x) e(\phi)(\tau, x) d\tau] dx dt. \end{aligned}$$

Now an integration by parts yields

$$\lim_{\epsilon \rightarrow 0} \int_0^T \int_U \operatorname{div} T \cdot \phi dx dt = \quad (6.17)$$

$$\int_0^T \int_U \operatorname{div} [T_1 e(\mathbf{u}_0) + T_2 e(\partial_t \mathbf{u}_0) + \int_0^t T_3(t - \tau, \cdot) e(\mathbf{u}_0)(\tau, \cdot) d\tau] \cdot \phi dx dt$$

for any test function  $\phi(x, t)$ . The expression in brackets defines the effective stress tensor that combines elastic and memory effects. The term  $T_1 e(\mathbf{u}_0)$  represents the elastic part of the overall stress, while the other two terms describe the viscoelastic part. Using (6.17) we can write down the effective equations as

$$\rho_0 \partial_t \mathbf{u}_0 - \operatorname{div} (T_1 e(\mathbf{u}_0) + T_2 e(\partial_t \mathbf{u}_0) + \int_0^t T_3(t - \tau, \cdot) e(\mathbf{u}_0)(\tau, \cdot) d\tau) = f. \quad (6.18)$$

Clearly, when  $T_1$  is much smaller than  $T_2$  our equations are fluid-like, and if  $T_2$  is negligible compared to  $T_1$  we obtain a solid-like effective model.

## 7 Mathematical Remarks.

### 7.1 Geometry.

Without loss of generality we assume that  $U \subset \{x : \alpha \leq |x| \leq \beta\}$  for some positive  $\alpha, \beta$ . We also assume that  $\theta(x, \epsilon)$  is a smooth function of its arguments for  $0 < \epsilon \leq 1$ .

Given  $x, y$  such that  $x \cdot y \neq 0$  we define

$$t = y - \frac{x \cdot y}{|x|^2} x, \quad (7.1)$$

$$\epsilon_x = \frac{|x|^2}{x \cdot y},$$

and

$$w = \epsilon_x t.$$

Note that  $t \cdot x = 0$ .

**Proposition 7.1** *The following are equivalent.*

i) *There exists a smooth function  $Q(x, y)$  such that*

$$Q(x, \frac{x}{\epsilon}) = \theta(x, \epsilon), \quad (7.2)$$

$$|\nabla_x Q| \leq C, \quad (7.3)$$

$$|\nabla_y Q| \leq C, \quad (7.4)$$

$$|\nabla_y Q(x, \frac{x}{\epsilon}) - \frac{x}{|x|^2} x \cdot \nabla_y Q(x, \frac{x}{\epsilon})| \leq C\epsilon. \quad (7.5)$$

ii) *The function  $\theta(x, \epsilon)$  satisfies*

$$\epsilon^2 |\partial_\epsilon \theta| \leq C, \quad (7.6)$$

$$|\nabla_x \theta + \epsilon \partial_\epsilon \theta \frac{x}{|x|^2}| \leq C. \quad (7.7)$$

*Proof.* i)  $\Rightarrow$  ii). Differentiating both sides of  $\theta(x, \epsilon) = Q(x, \frac{x}{\epsilon})$  we find

$$\begin{aligned} \nabla_x \theta(x, \epsilon) &= \nabla_x Q(x, \frac{x}{\epsilon}) + \frac{1}{\epsilon} \nabla_y Q(x, \frac{x}{\epsilon}), \\ \partial_\epsilon \theta(x, \epsilon) &= -\frac{1}{\epsilon^2} x \cdot \nabla_y Q(x, \frac{x}{\epsilon}). \end{aligned}$$

The second equation shows that (7.4) implies (7.6). Then, combining the equations we obtain

$$\nabla_x \theta(x, \epsilon) + \epsilon \partial_\epsilon \theta \frac{x}{|x|^2} = \nabla_x Q(x, \frac{x}{\epsilon}) + \frac{1}{\epsilon} (\nabla_y Q - \frac{x}{|x|^2} x \cdot \nabla_y Q(x, \frac{x}{\epsilon})).$$

This shows that (7.3) and (7.5) imply (7.7).

ii)  $\Rightarrow$  i). First we extend  $\theta(x, \epsilon)$  from  $U \times (0, 1]$  to  $U \times \mathbf{R}^+$  so that the extension, still denoted by  $\theta(x, \epsilon)$  is smooth, satisfies (7.6), (7.7), and equals zero when  $\epsilon \geq 2$ . Then we extend  $\theta$  from  $U \times \mathbf{R}^+$  to  $U \times \mathbf{R} \setminus \{0\}$  by setting  $\theta(x, -\epsilon) = \theta(x, \epsilon)$ . Fix  $k$  such that  $2k \leq \alpha$ , where  $\alpha$  is the constant in the bounds for  $|x|$  from the beginning of this Subsection. Let  $h(w)$  be a smooth cut-off function supported in the unit ball centered at zero, and such that  $h = 1$  in a ball of radius  $1/2$ . We define

$$Q(x, y) = \theta(x, \epsilon_x) h(\epsilon_x t) \quad (7.8)$$

when  $|x \cdot y| \geq k$  and  $Q = 0$  otherwise, with  $\epsilon_x$  and  $t$  defined above. Note that  $\epsilon_x < 2$  when  $k$  is chosen as above. Differentiating (7.8) we find

$$\nabla_x Q(x, y) = h(\epsilon_x t) \nabla_x \theta(x, \epsilon_x) + h(\cdot) \partial_\epsilon \theta(\cdot) \frac{x}{|x|^2} \epsilon_x - \quad (7.9)$$

$$\begin{aligned}
& h(\cdot)\partial_\epsilon\theta(\cdot)t\epsilon_x^2 + \theta(\cdot)(t \cdot \nabla_w h(\cdot)\frac{t}{|x|^2}\epsilon_x^2 + t \cdot \nabla_w h\frac{x}{|x|^2}\epsilon_x + \\
& \quad x \cdot \nabla_w h(\cdot)\frac{t}{|x|^2}\epsilon_x + x \cdot \nabla_w h(\cdot)\frac{x}{|x|^2} - \nabla_w h(\cdot)), \\
\nabla_y Q(x, y) &= -\partial_\epsilon\theta(\cdot)h(\cdot)\frac{x}{|x|^2}\epsilon_x^2 - \theta(\cdot)(t \cdot \nabla_w h(\cdot)\frac{t}{|x|^2}\epsilon_x^2 + t \cdot \nabla_w h\frac{x}{|x|^2}\epsilon_x + (7.10) \\
& \quad x \cdot \nabla_w h(\cdot)\frac{t}{|x|^2}\epsilon_x + x \cdot \nabla_w h(\cdot)\frac{x}{|x|^2} - \nabla_w h(\cdot)\epsilon_x).
\end{aligned}$$

From (7.10) we deduce

$$\begin{aligned}
\nabla_y Q(x, y) - \frac{x}{|x|^2}x \cdot \nabla_y Q(x, y) &= (7.11) \\
\theta(\cdot)\epsilon_x(\nabla_w h(\cdot) - t \cdot \nabla_w h(\cdot)\frac{t}{|x|^2}\epsilon_x - x \cdot \nabla_w h(\cdot)\frac{x}{|x|^2}).
\end{aligned}$$

Since derivatives of  $h$  are compactly supported, and  $|x|$  is bounded from above and below, we see that (7.11) implies (7.5). The same reasoning shows that (7.10) together with (7.6) implies (7.4). Finally, examination of (7.9) shows that (7.6) and (7.7) imply (7.3).

## 7.2 Small Compressibility.

To make the map  $\chi$  well-defined, we need to assume that divergence  $\mathbf{v}$  is bounded pointwise. Verification of this condition for compressible fluids may be difficult. Instead, we could require that the density  $\rho(x, t)$  satisfies

$$\rho_0 e^{-\gamma} \leq \rho \leq \rho_0 e^\gamma$$

for some  $\gamma > 0$ . As shown by Hoff [8], these bounds can be obtained provided the initial conditions are small enough. Thus one can replace small compressibility assumption by a smallness assumption on the initial conditions. We are planning to address this matter in a future publication.

When the initial velocity tends to zero, convergence to linearized system equivalent to (5.1), (5.2) was proved by Lions and Masmoudi [11]. In this paper the size of initial conditions is fixed, and the small parameter characterizes length scale separation.

## 7.3 Linearization.

Our goal is to use energy estimates together with the Poincaré inequality (3.3) to show that  $\mathbf{w}$  is small. The strategy is to take regularity of  $\bar{\mathbf{v}}$  for granted and then prove that  $\mathbf{w}$  is small in  $L^2(0, T; H^1(U))$ . Let us list regularity requirements on  $\bar{\mathbf{v}}$ .

We postulate that there is  $C$  independent of  $\epsilon$  and such that

$$i) \|\bar{\mathbf{v}}\|_{L^\infty([0, T] \times V^\epsilon)} \leq C;$$



$$\text{ii) } \int_0^T \|\bar{\mathbf{v}}\|_{H^1(V^\epsilon)}^2(t) dt \leq C.$$

The small compressibility assumption is used in its most relaxed form, namely

iii) there is  $\gamma > 0$  independent of  $\epsilon$  such that

$$\rho_0 e^{-\gamma} \leq \rho(x, t) \leq \rho_0 e^\gamma.$$

Moreover, we take for granted existence of the energy-type  $\epsilon$ -independent a priori bounds for  $\mathbf{v}$  in  $L^2(0, T; H_0^1(U))$ .

In what follows, we will need two technical results. First we obtain a parameter-dependent Poincaré inequality.

**Proposition 7.2** *Suppose that  $\mathbf{w} \in L^2(0, T; H^1(V^\epsilon))$  and  $\mathbf{w} = 0$  on  $U \cap \partial V^\epsilon$ . Under assumption 3.1, the inequality*

$$\int_U |\mathbf{w}(x, t)|^2 dx \leq L\epsilon^2 \int_U |\nabla \mathbf{w}(x, t)|^2 dx \quad (7.12)$$

holds for almost all  $t \in [0, \infty]$ ,  $L$  being the constant from (3.3).

*Proof.* Extend  $\mathbf{w}$  by zero to all of  $U$ . Then, rescaling (3.3) we obtain (7.12) for all cubes  $C_k^\epsilon$  with  $C$  independent of  $k$ . Summing up over  $C_k$  finishes the proof.

We will also need another result obtained by straightforward computation.

**Lemma 7.1** *Suppose that  $\rho, \mathbf{u}$  are sufficiently smooth and*

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = f,$$

Then

$$(\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u})) \cdot \mathbf{u} = f \frac{|\mathbf{u}|^2}{2} + \partial_t(\rho \frac{|\mathbf{u}|^2}{2}) + \operatorname{div}(\rho \mathbf{u} \frac{|\mathbf{u}|^2}{2}). \quad (7.13)$$

**Theorem 7.1** *Suppose that conditions i), ii) iii) are satisfied. For  $\epsilon, \gamma$  small enough,*

$$\int_0^T \int_U |\mathbf{w}|^2(x, t) dx \leq C\epsilon^2, \quad (7.14)$$

where  $C$  depends on  $\gamma, \mu, L$ , the constant in i), ii), and the a priori bound on  $\mathbf{v}$ .

*Proof.*

Multiplying (5.3) by  $\mathbf{w}$  and using conservation of mass and (7.13) we find formally

$$[\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v})] \cdot \mathbf{w} = \partial_t(\rho \frac{|\mathbf{w}|^2}{2}) + \operatorname{div}(\rho \bar{\mathbf{v}}) \frac{|\mathbf{w}|^2}{2} + \quad (7.15)$$

$$\operatorname{div}(\rho(\bar{\mathbf{v}} \otimes \mathbf{w})\mathbf{w} + \rho(\mathbf{w} \otimes \bar{\mathbf{v}}) \cdot \mathbf{w} - 2\rho\bar{\mathbf{v}}\frac{|\mathbf{w}|^2}{2}) + [\partial_t(\rho\bar{\mathbf{v}}) + \operatorname{div}(\rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}})] \cdot \mathbf{w},$$

Furthermore,

$$\operatorname{div}(\rho\bar{\mathbf{v}})\frac{|\mathbf{w}|^2}{2} = \operatorname{div}(\rho\bar{\mathbf{v}}\frac{|\mathbf{w}|^2}{2}) - \rho(\bar{\mathbf{v}} \cdot \nabla)\mathbf{w} \cdot \mathbf{w}, \quad (7.16)$$

and

$$\operatorname{div}(\rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}}) \cdot \mathbf{w} = \operatorname{div}(\rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}}\mathbf{w}) - \rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \cdot \nabla\mathbf{w}. \quad (7.17)$$

Combining the last two equalities with (7.15) we obtain

$$[\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v} \otimes \mathbf{v})] \cdot \mathbf{w} = \quad (7.18)$$

$$\partial_t(\rho\frac{|\mathbf{w}|^2}{2}) + \partial_t(\rho\bar{\mathbf{v}}) \cdot \mathbf{w} - \rho(\bar{\mathbf{v}} \cdot \nabla)\mathbf{w} \cdot \mathbf{w} - \rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \cdot \nabla\mathbf{w} + \operatorname{div} M,$$

where  $M$  denotes a quantity with zero trace on the boundary of  $V^\epsilon$ .

Similarly,

$$[\mu\Delta\mathbf{v} + \xi\nabla\operatorname{div}\mathbf{v} - \nabla P] \cdot \mathbf{w} = -\mu|\nabla\mathbf{w}|^2 - \xi(\operatorname{div}\mathbf{w})^2 + \quad (7.19)$$

$$\mu\Delta\bar{\mathbf{v}} \cdot \mathbf{w} + \xi\nabla\operatorname{div}\bar{\mathbf{v}} \cdot \mathbf{w} - \nabla P \cdot \mathbf{w} + \operatorname{div}(\mu\nabla\frac{|\mathbf{w}|^2}{2} + \xi\mathbf{w}\operatorname{div}\mathbf{w}).$$

Combining the results with (5.3) we obtain the formal energy identity

$$\partial_t(\rho\frac{|\mathbf{w}|^2}{2}) + \mu|D\mathbf{w}|^2 + \xi(\operatorname{div}\mathbf{w})^2 + \operatorname{div} M = \quad (7.20)$$

$$-[\partial_t(\rho\bar{\mathbf{v}}) - \mu\Delta\bar{\mathbf{v}} + \xi\nabla\operatorname{div}\bar{\mathbf{v}} - \nabla P] \cdot \mathbf{w} + f \cdot \mathbf{w} + \rho(\bar{\mathbf{v}} \cdot \nabla)\mathbf{w} \cdot \mathbf{w} + \rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \cdot \nabla\mathbf{w}.$$

From the equation (5.3) for  $\bar{\mathbf{v}}$ ,

$$\partial_t(\rho\bar{\mathbf{v}}) - \mu\Delta\bar{\mathbf{v}} - \xi\nabla\operatorname{div}\bar{\mathbf{v}} - \nabla P = \quad (7.21)$$

$$f - \partial_t\rho_1\bar{\mathbf{v}} - \rho_1\partial_t\bar{\mathbf{v}} - \nabla(P - P_0),$$

where  $P_0$  is the linearized pressure defined by

$$P_0 = a(\rho_0 + \rho_1) = a\rho_0(1 - \int_0^t \operatorname{div}\bar{\mathbf{v}}(x, \tau)d\tau).$$

Substitution into the energy identity yields

$$\partial_t(\rho\frac{|\mathbf{w}|^2}{2}) + \mu|D\mathbf{w}|^2 + \xi(\operatorname{div}\mathbf{w})^2 + \operatorname{div} M = \quad (7.22)$$

$$\partial_t\rho_1\bar{\mathbf{v}} + \rho_1\partial_t\bar{\mathbf{v}} + \nabla(P - P_0) + \rho(\bar{\mathbf{v}} \cdot \nabla)\mathbf{w} \cdot \mathbf{w} + \rho\bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \cdot \nabla\mathbf{w},$$

where  $\rho_1$  is the perturbation of the density from the previous section, and  $M$  is zero on  $\partial V^\epsilon$ . We integrate (7.22) over  $V^\epsilon$  and estimate various terms on the right. First, we estimate the integral containing pressure.

$$\int_{V^\epsilon} \nabla(P_0 - P) \cdot \mathbf{w} dx = - \int_{V^\epsilon} (P_0 - P)\operatorname{div}\mathbf{w} dx. \quad (7.23)$$

Writing

$$\rho^\delta = (\rho^\delta + (\delta - 1)\rho_0^\delta - \delta\rho\rho_0^{\delta-1}) + \delta\rho_0^{\delta-1}\rho, \quad (7.24)$$

we estimate the quantity

$$\rho^\delta - [\delta\rho\rho_0^{\delta-1} - (\delta - 1)\rho_0^\delta].$$

The expression in brackets is the equation of the tangent to the graph of  $y = \rho^\delta$  passing through  $\rho = \rho_0$ . Using convexity of  $\rho^\delta$  and assumption iii) we obtain

$$0 \leq \rho^\delta - [\delta\rho\rho_0^{\delta-1} - (\delta - 1)\rho_0^\delta] \leq \max \{(\rho_0^\delta e^{\gamma\delta} + (\delta - 1)\rho_0^\delta - \delta e^\gamma \rho_0^\delta), (\rho_0 e^{-\gamma\delta} + (\delta - 1)\rho_0^\delta - \delta\rho_0^\delta e^{-\gamma})\}.$$

Applying Taylor theorem we deduce from this that there is  $C(\gamma_0)$  such that

$$0 \leq \rho^\delta + (\delta - 1)\rho_0^\delta - \delta\rho\rho_0^{\delta-1} \leq C(\gamma_0)\rho_0^\delta\gamma^2(\delta^2 - \delta) \quad (7.25)$$

for all  $\gamma \in [0, \gamma_0]$ . Returning to (7.24) we see that

$$\begin{aligned} P &= a\rho^\delta \leq C(\gamma_0)a\rho_0^\delta\gamma^2(\delta^2 - \delta) + a\delta\rho_0^{\delta-1}\rho = \\ &C(\gamma_0)a\rho_0^\delta\gamma^2(\delta^2 - \delta) + a\delta\rho_0^{\delta-1}(\rho_0 + \rho_1). \end{aligned} \quad (7.26)$$

Dropping the constant term we find

$$\int_{V^\epsilon} \nabla(P - P_0) \cdot \mathbf{w} dx \leq \quad (7.27)$$

$$\begin{aligned} &[C(\gamma_0, \delta, \rho_0)\gamma^2 + a|(\delta\rho_0^{\delta-1} - \rho_0)| \sup |\rho_1|] \|\nabla \mathbf{w}\|_{L^2(U)} \leq \\ &C(\gamma)(\kappa \|\nabla \mathbf{w}\|_{L^2(U)}^2 + C(\kappa)), \end{aligned}$$

where we used Young's inequality. It should be noted that  $C(\gamma) \rightarrow 0$ ,  $|\rho_1| \rightarrow 0$  when  $\gamma \rightarrow 0$ , and  $\kappa > 0$  can be chosen arbitrarily small.

Next we estimate the term  $\partial_t \rho_1 \bar{\mathbf{v}}$ . From the mass balance equation,

$$\partial_t \rho_1 = -\rho_0 \operatorname{div} \mathbf{v} - \operatorname{div}(\rho_1 \mathbf{v}).$$

Hence

$$\int_{V^\epsilon} \partial_t \rho_1 \bar{\mathbf{v}} \cdot \mathbf{w} dx = - \int_{V^\epsilon} \rho_0 (\operatorname{div} \mathbf{v}) \bar{\mathbf{v}} \cdot \mathbf{w} dx - \int_{V^\epsilon} \operatorname{div}(\rho_1 \mathbf{v}) \bar{\mathbf{v}} \cdot \mathbf{w} dx.$$

Making use of the estimate

$$\|\operatorname{div}(\rho_1 \mathbf{v})\|_{H^{-1}(V^\epsilon)} \leq \sup |\rho_1| \|\mathbf{v}\|_{L^2(V^\epsilon)},$$

and Young's inequality we obtain

$$\int_{V^\epsilon} \partial_t \rho_1 \bar{\mathbf{v}} \cdot \mathbf{w} dx \leq \quad (7.28)$$

$$C \|\bar{\mathbf{v}}\|_{L^\infty(V^\epsilon)} \left( \sup |\rho_1| \kappa \|\nabla \mathbf{w}\|_{L^2(V^\epsilon)}^2 + \sup |\rho_1| C(\kappa) \|\mathbf{v}\|_{L^2(V^\epsilon)}^2 + \kappa \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa) \|\mathbf{w}\|_{L^2(V^\epsilon)}^2 \right).$$

The term  $\rho_1 \partial_t \bar{\mathbf{v}} \cdot \mathbf{w}$  is estimated as follows. First, from the weak formulation of the momentum equation for  $\bar{\mathbf{v}}$  we obtain a bound on  $H^{-1}$ -norm of  $\partial_t \bar{\mathbf{v}}$ :

$$\|\partial_t \bar{\mathbf{v}}\|_{H^{-1}(V^\epsilon)} \leq C(\|\bar{\mathbf{v}}\|_{H^1(V^\epsilon)}, \rho_0, f, t). \quad (7.29)$$

Hence,

$$\begin{aligned} \left| \int_{V^\epsilon} \rho_1 \partial_t \bar{\mathbf{v}} \cdot \mathbf{w} dx \right| &\leq C \sup |\rho_1| \|\nabla \mathbf{w}\|_{L^2(U)} \leq \\ &C \sup |\rho_1| (\kappa \|\nabla \mathbf{w}\|_{L^2(V^\epsilon)}^2 + C(\kappa)), \end{aligned} \quad (7.30)$$

Combining this inequality and (7.28) we find

$$\begin{aligned} \int_{V^\epsilon} \partial_t(\rho_1 \bar{\mathbf{v}}) \cdot \mathbf{w} dx &\leq C(\sup |\rho_1| \kappa \|\nabla \mathbf{w}\|_{L^2(U)} + \\ &\sup |\rho_1| C(\kappa) \|\mathbf{v}\|_{L^2(V^\epsilon)}^2 + \\ &\kappa \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa) \|\mathbf{w}\|_{L^2(V^\epsilon)}^2 + C(\kappa)). \end{aligned} \quad (7.31)$$

It is important to observe that  $\kappa$  can be chosen arbitrarily small and  $|\rho_1|$  can be made arbitrarily small by choosing  $\gamma$  small enough.

Next, using Young's inequality, we write

$$\begin{aligned} \int_{V^\epsilon} \rho \bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \cdot \nabla \mathbf{w} dx &\leq \kappa \int_{V^\epsilon} |\nabla \mathbf{w}|^2 dx + C(\kappa) \int_{V^\epsilon} |\rho \bar{\mathbf{v}} \otimes \bar{\mathbf{v}}|^2 dx \leq \\ &\kappa \|\nabla \mathbf{w}\|_{L^2(U)}^2 + C(\kappa) \end{aligned} \quad (7.32)$$

where  $\kappa$  can be chosen arbitrary small. The last remaining term is estimated by Young's inequality:

$$\int_{V^\epsilon} \rho(\bar{\mathbf{v}} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} dx \leq \|\rho \bar{\mathbf{v}}\|_{L^\infty(V^\epsilon)} (\kappa \|\nabla \mathbf{w}\|_{L^2(U)}^2 + C(\kappa) \|\mathbf{w}\|_{L^2(U)}^2). \quad (7.33)$$

Combining estimates (7.27), (7.28), (7.30), (7.32) and (7.33), we see that the right hand side of the energy identity is bounded by

$$\begin{aligned} &C(\kappa \|\nabla \mathbf{w}\|_{L^2(U)}^2 + C(\kappa) \|\mathbf{w}\|_{L^2(U)}^2) + \\ &\kappa \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa) \|\mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa). \end{aligned} \quad (7.34)$$

Thus we have

$$\begin{aligned} d_t \int_U \rho \frac{|\mathbf{w}|^2}{2} dx + \mu \int_U |\nabla \mathbf{w}|^2 dx &\leq \\ &C(\kappa \|\nabla \mathbf{w}\|_{L^2(U)}^2 + C(\kappa) \|\mathbf{w}\|_{L^2(U)}^2) + \\ &\kappa \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa) \|\mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa). \end{aligned} \quad (7.35)$$

Choosing  $\kappa$  sufficiently small, we deduce that there is a positive  $\nu$  such that

$$d_t \int_U \rho \frac{|\mathbf{w}|^2}{2} dx + \nu \int_U |\nabla \mathbf{w}|^2 dx \leq \quad (7.36)$$

$$C(C(\kappa) \|\mathbf{w}\|_{L^2(U)}^2 + \kappa \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa) \|\mathbf{v}\|_{L^2(V^\epsilon)}^2 + C(\kappa)).$$

Estimating the second integral on the left from below by Poincaré inequality from Proposition 7.2 we find

$$d_t \int_U \rho \frac{|\mathbf{w}|^2}{2} dx + \left(\frac{\nu}{\epsilon^2 L} - C(\kappa)\right) \int_U |\mathbf{w}|^2 dx \leq \quad (7.37)$$

$$C(\kappa) \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)} + C(\kappa) \|\mathbf{v}\|_{L^2(V^\epsilon)} + C(\kappa).$$

From now on, we fix  $\kappa$ . When  $\epsilon \rightarrow 0$ ,

$$\frac{\nu}{\epsilon^2 L} - C(\kappa) \geq \frac{1}{\epsilon^2} K,$$

where  $K > 0$  depends on  $\kappa, \mu$  and  $L$ . Let us denote  $W(t) = \int_0^t \int_U \rho |\mathbf{w}|^2(x, \tau) dx d\tau$ . Integrating (7.37) in time we obtain

$$d_t W + \frac{1}{\epsilon^2} \frac{K}{b} W \leq C_1 \int_0^t \|\nabla \mathbf{v}\|_{L^2(V^\epsilon)}(\tau) d\tau + C_2 \int_0^t \|\mathbf{v}\|_{L^2(V^\epsilon)}(\tau) d\tau + C_3 t,$$

where  $b = \rho_0 e^\gamma$ . By assumption, both integrals on the right are bounded independent of  $\epsilon$ , so that there is  $\tilde{C}$  independent of  $\epsilon$  such that the right hand side of the last inequality is bounded by  $\tilde{C} + C_3 t$ . Multiplying both sides by

$$e^{\frac{K}{\epsilon^2 b} t},$$

integrating and using  $W(0) = 0$  we arrive at the inequality

$$W(t) \leq \frac{\epsilon^2 b}{K} (\tilde{C} + C_3 t).$$

Since  $\rho$  is bounded by  $\rho_0 e^{-\gamma}$  from below, this implies

$$\int_0^T \int_U |w(x, t)|^2 dx dt \leq \epsilon^2 \frac{e^{2\gamma}}{K} (\tilde{C} + C_3 T),$$

which proves the theorem.

The theorem implies that as  $\epsilon \rightarrow 0$ ,  $\mathbf{v}$  tends to  $\bar{\mathbf{v}}$  in  $L^2(0, T; L^2(U))$ .

## 7.4 Averaged Equations.

To justify passing to the limit  $\epsilon \rightarrow 0$  in (6.10) and (6.14)-(6.16) we note that  $\mathbf{u}(t, x, \epsilon)$  is bounded in  $L^2(0, T; H^1(U))$  independent of  $\epsilon$ , provided initial conditions are independent of  $\epsilon$ . Moreover, standard considerations lead to the bound on the time derivative

$$\| \partial_t \mathbf{u} \|_{L^2(0, T; H^{-1}(V^\epsilon))} \leq C$$

with  $C$  independent of  $\epsilon$ . Applying J.L. Lions's compactness theorem we deduce existence of a subsequence that converges to some  $\mathbf{u}_0$  in  $L^2(0, T; L^2(U))$  as  $\epsilon \rightarrow 0$ . On the other hand,  $\operatorname{div}_x (B_1 - B_2)$  is bounded in  $L^2(0, T; L^2(U))$  uniformly in  $\epsilon$ . This follows from uniform in  $\epsilon$  estimates

$$\| e_y(N^{pq})(x, \frac{x}{\epsilon}) \|_{L^2(U)} \leq C,$$

$$\| e_y(M_0^{pq})(x, \frac{x}{\epsilon}) \|_{L^2(U)} \leq C,$$

and

$$\| e_y(M^{pq})(x, \frac{x}{\epsilon}) \|_{L^2(0, T; L^2(U))} \leq C,$$

which are obtained from the local problems. These energy-type estimates can be obtained by standard methods applied to the local problems in the domain  $U_{1/\epsilon} = \{y : \epsilon y \in U\}$ .

The bound on  $\operatorname{div}_x (B_1 - B_2)$  yields existence of a subsequence converging weakly in  $L^2(0, T; L^2(U))$ . Product of a weakly and strongly convergent sequences converges to the product of limits.

Finally, we analyze the inertial term in the microscopic linearized equations (6.1). Since  $\mathbf{u}$  converges strongly to  $\mathbf{u}_0$ , the product  $\rho_0 \mathbf{u}$  converges to  $\rho_0 \mathbf{u}_0$  in  $L^2(0, T; L^2(U))$ . Then  $\rho_0 \partial_t^2 \mathbf{u} = \partial_t^2 (\rho_0 \mathbf{u})$  converges to  $\partial_t^2 \rho_0 \mathbf{u}_0$  in the distribution sense.

This and computations leading to (6.17) prove the following

**Theorem 7.2** *Suppose the boundary and initial conditions for the linearized equations (6.1) are independent of  $\epsilon$ . Then there exists a sequence  $\epsilon_n \rightarrow 0$  such that the corresponding sequence  $\mathbf{u}(\cdot, \epsilon_n)$  converges in the distribution sense to the solution  $\mathbf{u}_0$  of the effective equations*

$$\rho_0 \partial_t^2 \mathbf{u}_0 - \operatorname{div} ((T_1 e(\mathbf{u}_0) + T_2 e(\partial_t \mathbf{u}_0) + \int_0^t T_3(t - \tau, \cdot) e(\mathbf{u}_0)(\tau, \cdot) d\tau) = f, \quad (7.38)$$

where the tensors  $T_1, T_2, T_3$  are given by (6.14)-(6.16).

## References

- [1] A .Yu. Beliaev and S.M. Kozlov, *Darcy Equation for Random Porous Media*, Communications in Pure and Applied Mathematics, v. 69, (1996), 1-34.

- [2] M. A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid. Part I: Lower frequency range, and Part II: Higher frequency range*, J. Acoust. Soc. Amer. **28** (1956), 168-178, and 179-191.
- [3] M. A. Biot, *Mechanics of deformation and acoustic propagation in porous media*, J. Applied Physics **33** (1962), 1482-1498.
- [4] M. J. Buckingham, *Wave propagation, stress relaxation, and grain-to-grain shearing in saturated, unconsolidated marine sediments*, J. Acoust. Soc. Am. **108** (6), (2000), 2796-2815.
- [5] R. Burridge and J. B. Keller, *Poroelasticity equations derived from microstructure*, J. Acoust. Soc. Amer., **70** (1981), 1140-1146.
- [6] R. P. Gilbert and A. Mikelic, *Homogenizing the acoustic properties of the seabed: Part I*, Nonlinear Analysis, **40** (2000) 185-212.
- [7] R. P. Gilbert and A. Panchenko, *Acoustics of a stratified poroelastic composite*, Zeitschrift für Analysis und ihre Anwendungen, **18**(4) (1999) 977-1001.
- [8] D. Hoff, *Discontinuous solutions of the Navier-Stokes equations for multi-dimensional flows of heat-conducting fluids*, Arch. Rational Mech. Anal. **139** (1997), 303-354.
- [9] J. T. Jenkins and S. B. Savage, *A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles*, Journal of Fluid Mechanics, v. 130 (1983), 187-202.
- [10] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer, Berlin (1994).
- [11] P.-L. Lions and N. Masmoudi, *Incompressible limit for a viscous compressible fluid*, J. Math. Pures Appl., **77** (1998), 585-627.
- [12] T. Levy, *Fluids in porous media and suspensions in: Homogenization Techniques in Composite Media*, Springer, Berlin (1987).
- [13] E. Sanchez-Palencia, *Non-Homogenous Media and Vibration Theory*, Springer, Berlin (1980).
- [14] L. Tartar, *Cours Peccot au Collège de France*, (1977), (Preprint).
- [15] L. Tartar, *Appendix in Non-Homogenous Media and Vibration Theory*, Springer, Berlin (1980).
- [16] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York (1989).