Numerical Computation of Rectangular Bivariate and Trivariate Normal and t Probabilities

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Abstract

Algorithms for the computation of bivariate and trivariate normal and t probabilities for rectangles are reviewed. The algorithms use numerical integration to approximate transformed probability distribution integrals. A generalization of Plackett's formula is derived for bivariate and trivariate t probabilities. New methods are described for the numerical computation of bivariate and trivariate t probabilities. Test results are provided, along with recommendations for the most efficient algorithms for single and double precision computations.
Key Words: distribution, bivariate normal, trivariate normal, bivariate t , trivariate t, Plackett formula

1 Introduction

Bivariate and trivariate probability distribution computations are needed for many statistics applications. Although reliable, efficient and accurate algorithms for univariate probabilities have been available for some time, high quality algorithms for bivariate and trivariate probability distribution computations have only more recently started to become available. There is a need for these algorithms as components in statistical computation libraries and packages. There is also an increasing need for these algorithms as a means for efficiently computing good bounds for multivariate probabilities. These bounds can then be used to provide more efficient computation methods for multivariate probability computations (see Gassmann, 2000, and Genz, Bretz, and Hochberg, 2003).

There are now many algorithms available for computation of bivariate normal (BVN) probabilities, but the quality of these algorithms has significant variation. In a comparative study of these algorithms by Terza and Welland (1990), Divgi's method (1979) was found to outperform a number of other methods for low accuracy work. Shortly after that paper appeared, Drezner and Wesolowsky (1990) presented a simple method for single precision work that used less computation time than Divgi's algorithm. Recently, after a study of different algorithms, Patefield and Tandy (2000) developed a hybrid double precision algorithm.

Until very recently, there were no accurate and efficient algorithms available for the general trivariate normal (TVN) problem. Schervish's algorithm (1984) was the first published numerical algorithm, but this algorithm was developed for the general multivariate normal problem. This algorithm was followed by algorithms developed by Cox and Wermuth (1991), Wang and Kennedy (1992), and Drezner (1992, 1994). The recent paper by Gassmann (2000) studied these algorithms and made some recommendations for algorithms for efficient and reliable TVN computations. The primary algorithm for bivariate t (BVT) probabilities was developed by Dunnett and Sobel (1954). There are several general algorithms available for multivariate t probability computations (see the review by Genz and Bretz, 2002) but these algorithms usually cannot efficiently provide high accuracy results. There are currently no published specialized algorithms for trivariate t (TVT) probabilities.

In this paper there is a brief discussion of some modifications to the algorithm of Drezner and Wesolowsky which provide a double precision BVN algorithm in a simpler form than the algorithm provided by Patefield and Tandy (2000). This is used as the basis for some TVN algorithms, and some comparisons of different TVN algorithms are given, along with some discussion of extensions of Gassmann's work for reliable double precision computations. These considerations provide some background for work on algorithms for bivariate and trivariate t probability computations. The main contributions of this paper are some generalizations of Plackett's (1954) formulas which provide the basis for some new algorithms for BVT and TVT probabilities. There is a discussion of test results for the new algorithms. Fortran software for BVN, BVT, TVN and TVT double precision implementations is available from the author's website (www.math.wsu.edu/faculty/genz/homepage, in TVPACK).

2 Bivariate Normal Probabilities

2.1 The Standard BVN Problem

The standard bivariate normal distribution function is given by

$$\Phi(\mathbf{b},\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} e^{-(x^2-2\rho xy+y^2)/(2(1-\rho^2))} dydx,$$

where $\mathbf{b} = (b_1, b_2)$. Early work on BVN computations (see Andel, 1974, and Terza and Welland, 1990) studied the bivariate normal probability defined by

$$L(h,k,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{h}^{\infty} \int_{k}^{\infty} e^{-(x^2-2\rho xy+y^2)/(2(1-\rho^2))} dydx, \tag{1}$$

which is related to the standard bivariate normal distribution function by $\Phi(\mathbf{b},\rho) = L(-b_1,-b_2,\rho)$. In this section the discussion will be focused on methods for the computation of $L(h,k,\rho)$, in order to allow consistent references to earlier work. Drezner and Wesolowsky (1990) studied the formula

$$L(h,k,\rho) = \Phi(-h)\Phi(-k) + \frac{1}{2\pi} \int_0^{\rho} \frac{1}{\sqrt{1-r^2}} e^{-\frac{h^2+k^2-2rhk}{2(1-r^2)}} dr,$$

and used numerical integration for computation of BVN probabilities. The formula derived by Plackett (1954) for the correlation coefficient partial derivative of the bivariate normal distribution can be written

$$\frac{\partial L(h,k,r)}{\partial r} = \frac{e^{-\frac{h^2-2rhk+k^2}{2(1-r^2)}}}{2\pi\sqrt{1-r^2}}. \tag{2}$$

The integration of this formula for r between 0 and ρ produces the formula studied by Drezner and Wesolowsky.

2.2 The Transformed BVN Problem

If the substitution $r = \sin\theta$ is used, then

$$L(h,k,\rho) = \Phi(-h)\Phi(-k) + \frac{1}{2\pi} \int_0^{\sin^{-1}(\rho)} e^{-\frac{h^2+k^2-2hk\sin(\theta)}{2\cos(\theta)^2}} d\theta, \tag{3}$$

but there is a singularity in the integrand when $r = 1$ with both formulas, and the Drezner and Wesolowsky fixed-rule numerical integration method loses accuracy when $|\rho| \simeq 1$. In order to avoid this singularity, Drezner and Wesolowsky integrated between ρ and $s = sign(\rho)$ rather than between 0 and ρ to obtain

$$L(h,k,\rho) = L(h,k,s) - \frac{s}{2\pi} \int_{\rho}^{s} \frac{1}{\sqrt{1-r^2}} e^{-\frac{h^2+k^2-2hkr}{2(1-r^2)}} dr,$$

where

$$L(h,k,s) = \begin{cases} \Phi(-max(h,k)) & \text{if s = 1} \\ max(0, \Phi(-h) - \Phi(k)) & \text{if s = -1} \end{cases}.$$

If the substitution $x = \sqrt{1-r^2}$ is used, then

$$L(h,k,\rho) = L(h,k,s) - \frac{s}{2\pi} \int_0^{\sqrt{1-\rho^2}} \frac{1}{\sqrt{1-x^2}} e^{-\frac{h^2+k^2-2shk\sqrt{1-x^2}}{2x^2}} dx. \tag{4}$$

2.3 Numerical Integration Results

The numerical integration of (4) using a low degree Gauss rule is very accurate when $|\rho| \simeq 1$, except when h is close to but not equal to sk. In order to avoid the problems when $h \simeq sk$, Drezner and Wesolowsky rewrote (4) as

$$L(h,k,\rho) = L(h,k,s) - \frac{s}{2\pi} \int_0^{\sqrt{1-\rho^2}} \frac{1}{\sqrt{1-x^2}} e^{-\frac{(h-sk)^2}{2x^2}} e^{-\frac{shk}{1+\sqrt{1-x^2}}} dx.$$

The integrand in this expression is smooth except for the term $e^{-\frac{(h-sk)^2}{2x^2}}$ that is the source of relatively large errors when h is close to sk. Using the Taylor expansion

$$\frac{1}{\sqrt{1-x^2}} e^{-\frac{shk}{1+\sqrt{1-x^2}}} = e^{-\frac{shk}{2}}(1+(4-shk)x^2/8)+O(x^4), \tag{5}$$

$L(h,k,\rho)$ becomes

$$\begin{aligned} L(h,k,\rho) &= L(h,k,s) - \frac{s}{2\pi}\int_0^{\sqrt{1-\rho^2}} e^{-\frac{(h-sk)^2}{2x^2}} e^{-\frac{shk}{2}}(1+(4-shk)x^2/8)dx \\ &- \frac{s}{2\pi}\int_0^{\sqrt{1-\rho^2}} e^{-\frac{(h-sk)^2}{2x^2}} \left(\frac{e^{-\frac{shk}{1+\sqrt{1-x^2}}}}{\sqrt{1-x^2}} - e^{-\frac{shk}{2}}(1+(4-shk)x^2/8) \right) dx. \end{aligned} \tag{6}$$

The Drezner and Wesolowsky algorithm computes the second integral numerically, and the integral of the term $e^{-\frac{(h-sk)^2}{2x^2}} e^{-\frac{shk}{2}}(1+(4-shk)x^2/8)$ analytically, using the formula

$$\int_0^a e^{-b^2/(2x^2)}(1+cx^2)dx = a(1-c(b^2-a^2)/3)e^{-b^2/2a^2} - b(1-cb^2/3)\sqrt{2\pi}\Phi(-b/a),$$

with $a=\sqrt{1-\rho^2}$, $b=|h-sk|$ and $c=(4-shk)/8$.

2.4 Hybrid Numerical Integration Algorithms

A hybrid single precision algorithm can use the following strategy: use (3) for $|\rho| \le 0.8$ and use (6) for $|\rho| \ge 0.8$. Extensive testing shows that the maximum absolute error for this hybrid algorithm is $2.5 \cdot 10^{-7}$, if a 5-point Gauss-Legendre integration rule (see Davis and Rabinowitz, 1984, pp. 95–101) is used. A quadruple precision BVN algorithm based on Owen's (1956) series method was used as a source of highly accuracy BVN values for all of the tests described in this section.

The present author investigated the use of more Gauss points to produce an algorithm that would have maximum errors near 10^{-16} (double precision). In order to avoid very high numbers of Gauss points when $|\rho| \simeq 1$, a higher order Taylor expansion can be used in (5). Adding one more term to this expansion yields

$$\frac{1}{\sqrt{1-x^2}} e^{-\frac{shk}{1+\sqrt{1-x^2}}} = e^{-\frac{shk}{2}}(1+(4-shk)x^2(1+(12-shk)x^2/16)/8)+O(x^6). \tag{7}$$

If this modification is incorporated into equation (6) then a 20-point Gauss-Legendre rule used with the modified equation (6) for $|\rho| \ge 0.925$, and used with equation (3) for $|\rho| < 0.925$ results in a maximum absolute error less than $5 \cdot 10^{-16}$ in double precision. A more efficient algorithm can be constructed by selecting Gauss rules with fewer points for the smaller values of $|\rho|$. It was found, for example, that if the algorithm is modified to use only six points for $|\rho| < 0.3$ and twelve points for $0.3 \le |\rho| < 0.75$ then the same level of accuracy could be maintained. These final modifications were incorporated into the Fortran implementation that is available from the author. Further refinements are obviously possible, at the expense of additional implementation complexity.

3 Trivariate Normal Probabilities

3.1 The Standard TVN Problem

Let the standard trivariate normal distribution function be defined by

$$\Phi(\mathbf{b},R) = \frac{1}{(2\pi)^{3/2}|R|^{1/2}} \int_{-\infty}^{b_1}\int_{-\infty}^{b_2}\int_{-\infty}^{b_3} e^{-\frac{\mathbf{x}^T R^{-1}\mathbf{x}}{2}} dx_3 dx_2 dx_1, \tag{8}$$

where $\mathbf{b}=(b_1,b_2,b_3)$ and $R=(\rho_{ij})$ is a correlation matrix. More general TVN problems, where the covariance matrices do not have unit diagonal entries, can always be rescaled to be problems where the covariance matrix is a correlation matrix.

Owen (1956) briefly discussed the evaluation of the trivariate integral and gave the formula for the standard trivariate normal integral in terms of the bivariate normal integral as:

$$\Phi(\mathbf{b},R) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b_1} e^{-x^2/2} F(x) dx, \tag{9}$$

where

$$F(x) = \Phi\left(\left(\frac{b_2 - \rho_{21} x}{(1-\rho_{21}^2)^{\frac{1}{2}}}, \ \frac{b_3 - \rho_{31} x}{(1-\rho_{31}^2)^{\frac{1}{2}}} \right), \ \ \frac{\rho_{32} - \rho_{31}\rho_{21}}{(1-\rho_{21}^2)^{\frac{1}{2}}(1-\rho_{31}^2)^{\frac{1}{2}}} \right),$$

and $\Phi(h, k, \rho)$ is the standard bivariate normal distribution function.

Two methods were tested by the present author for calculating (9) using numerical integration. The first method that was tested transforms (9) using $x = \Phi^{-1}(t)$, so that

$$\Phi(\mathbf{b}, R) = \int_0^{\Phi(b_1)} F(\Phi^{-1}(t))dt, \tag{10}$$

and then a Gauss-Legendre numerical integration rule is used for the interval $[0, \Phi(b_1)]$.

The second method that was tested, which was also considered by Drezner (1992) as a method for general multivariate normal distribution computations, uses modified Gauss-Hermite rules (see Steen, Byrne and Gelhard, 1969). These rules are suitable for integrals in the form $\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} f(x)dx$, so (9) needs to be transformed to a $[0, \infty]$ integral. Let $y = x - b_1$, and then

$$\Phi(\mathbf{b}, R) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-(y+b_1)^2/2} F(y + b_1)dy = \frac{e^{-b_1^2/2}}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-y^2/2 - y b_1} F(y + b_1)dy.$$

Now let $z = -y$, so that

$$\Phi(\mathbf{b}, R) = \frac{e^{-b_1^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-z^2/2 + z b_1} F(b_1 - z)dz. \tag{11}$$

It was found that both of these methods were often more accurate for a particular integration rule if the integration limits were arranged so that the shortest integration interval was associated with the outermost integration.

3.2 Plackett TVN Formulas

The formula derived by Plackett for a correlation coefficient partial derivative of the trivariate normal distribution can be written

$$\frac{\partial \Phi(\mathbf{b}, R)}{\partial \rho_{21}} = \frac{e^{-f_3(\rho_{21})/2}}{2\pi\sqrt{1-\rho_{21}^2}} \Phi(u_3(\rho_{21})), \tag{12}$$

where $f_3(r) = \frac{b_1^2 + b_2^2 - 2r b_1 b_2}{(1-r^2)}$ and

$$u_3(r) = \frac{b_3(1-r^2) - b_1(\rho_{31} - r\rho_{32}) - b_2(\rho_{32} - r\rho_{31})}{((1-r^2)(1-r^2-\rho_{31}^2-\rho_{32}^2+2r\rho_{31}\rho_{32}))^{\frac{1}{2}}}.$$

Formulas for the two other off-diagonal ρ_{ij} partial derivatives can be obtained from equation (12) with appropriate permutations of the b's and ρ's.

Integration of this formula can provide TVN formulas which involve only two-dimensional integrals. Two Plackett formula methods, studied by Gassmann (2000), are also considered here. The first method uses

$$\Phi(\mathbf{b}, R) = \Phi(\mathbf{b}, R^*) + \frac{1}{2\pi} \int_{\rho_{21}^*}^{\rho_{21}} \frac{e^{-f_3(r)/2}}{\sqrt{1-r^2}} \Phi(u_3(r))dr, \tag{13}$$

where $\rho_{21}^* = \rho_{31}\rho_{32} \pm \sqrt{(1-\rho_{31}^2)(1-\rho_{32}^2)}$ and

$$R^* = \begin{bmatrix} 1 & \rho_{21}^* & \rho_{31} \\ \rho_{21}^* & 1 & \rho_{32} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}.$$

The matrix R^* is singular, so $\Phi(\mathbf{b}, R^*)$ can be computed using univariate and bivariate distribution values only. For efficient computation, a permutation of the ρ's in R (along with corresponding b's), and the sign used for ρ_{21}^*, are chosen to minimize the integration interval width $|\rho_{21}^* - \rho_{21}|$.

The second Plackett formula method uses

$$\Phi(\mathbf{b}, R) = \Phi(\mathbf{b}, R') + \frac{1}{2\pi} \int_0^1 \left(\rho_{21} \frac{e^{-f_3(\rho_{21} t)/2}}{\sqrt{1-\rho_{21}^2 t^2}} \Phi(\hat{u}_3(t)) + \rho_{31} \frac{e^{-f_2(\rho_{31} t)/2}}{\sqrt{1-\rho_{31}^2 t^2}} \Phi(\hat{u}_2(t)) \right) dt, \tag{14}$$

where $f_2(r) = \frac{b_1^2+b_3^2-2rb_1b_3}{(1-r^2)}$,

$$R' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho_{32} \\ 0 & \rho_{32} & 1 \end{bmatrix},$$

$$\hat{u}_2(t) = \frac{b_2(1-\rho_{31}^2t^2) - b_1t(\rho_{21}-\rho_{31}\rho_{32}) - b_3(\rho_{32}-\rho_{31}\rho_{21}t^2)}{((1-\rho_{31}^2t^2)(1-\rho_{31}^2t^2-\rho_{21}^2t^2-\rho_{32}^2+2t^2\rho_{31}\rho_{21}\rho_{32}))^{\frac{1}{2}}},$$

and

$$\hat{u}_3(t) = \frac{b_3(1-\rho_{21}^2t^2) - b_1t(\rho_{31}-\rho_{21}\rho_{32}) - b_2(\rho_{32}-\rho_{31}\rho_{21}t^2)}{((1-\rho_{21}^2t^2)(1-\rho_{21}^2t^2-\rho_{31}^2t^2-\rho_{32}^2+2t^2\rho_{31}\rho_{21}\rho_{32}))^{\frac{1}{2}}}.$$

In this case, $\Phi(\mathbf{b}, R')$ is easily computed as $\Phi(b_1)\Phi((b_2, b_3), \rho_{32})$. For efficient numerical integration, a permutation of the ρ's and b's, is chosen to minimize $max(|\rho_{31}|, |\rho_{21}|)$. The integrals for both Plackett formula methods were transformed using $r = sin(\theta)$, in order to remove integrand denominator singularities. This second Plackett method (14) was the method that Drezner (1994) implemented.

3.3 TVN Algorithm Testing

The testing of the algorithms requires a source of highly accurate TVN probabilities was needed for arbitrary $\mathbf{b}$ and R. Initially, Schervish's MULNOR was tried, but a number of inconsistencies were found, including some clear discrepancies between some exactly known results and the output from MULNOR. Kennedy and Wang (1992) also reported erroneous results from MULNOR, although the version of MULNOR used for these tests (emailed from statlib, and supported by the statlib $\Phi(x)$) did not fail for those TVN cases reported as failures by Kennedy and Wang. All tests reported in the rest of this section used an algorithm based on equation (14), implemented in quadruple precision with an adaptive numerical integration algorithm (requested absolute accuracy set at 10^{-18}) combined with a quadruple precision BVN algorithm based on Owen's (1956) series method, for comparison with single and double precision results.

In order to carry out careful tests of the different algorithms, a large set of general problems was needed with variation in all of the six parameters in $\mathbf{b}$ and R. Let $R = CC^T$, with

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \cos(\pi\theta_1) & \sin(\pi\theta_1) & 0 \\ \cos(\pi\theta_2)cos(\pi\theta_3) & \cos(\pi\theta_2)sin(\pi\theta_3) & \sin(\pi\theta_2) \end{bmatrix}. \tag{15}$$

The first tests that were completed used $\theta_1 = 1/258, 17/258, ..., 257/258$, $\theta_2 = 1/258, 17/258, ..., 257/258$, $\theta_3 = 1/258, 17/258, ..., 257/258$, $b_1 = -5, -4, ..., 5$, $b_2 = -5, -4, ..., 5$, $b_3 = b_2, b_2+1, ..., 5$. A complete test with all of the different combinations of parameters checks more than 2 million values for $\Phi(\mathbf{b}, R)$, and includes problems with $|R|$ as small as $2 \cdot 10^{-8}$ The methods using equations (10-11), (13-14) were tested using an integration rule with a fixed number, K, of integration rule points for K = 6, 12, 24, and the results are given in the first three rows of Table 1.

The Table 1 (first three row) results are consistent with results reported by Gassmann. The Plackett formula methods (13-14) were the most accurate. None of the TVN tests described so far used b values that were approximately equal. This type of limit combination was what motivated the final modifications described in the previous section for the Drezner-Wesolowsky BVN algorithm. In order to investigate the sensitivity of the TVN algorithms to this type of problems, another test was completed. This test used the same θ's that were used for the Table 1 first three row results, but used $b_1 = -5, -4, ..., 5$, $b_2 = -5+\epsilon, -4+\epsilon, ..., 5+\epsilon$, $b_3 = b_2, b_2+1, ..., 5$, with $\epsilon = 10^{-2}$. The 24-point rule results from this test are reported in Table 1, where it can be seen that the Drezner-Plackett method has a significant increase in maximum error. Additional tests were completed with other (larger and smaller) ϵ's but the largest errors occurred near $\epsilon = 10^{-2}$.

The results suggest that a 24-point Gauss-Legendre integration rule could be used with TVN algorithms based on the equations (13) or (14). These algorithms would produce single precision accuracy for most problems. The equation (13) algorithm appears to be less sensitive to the subtractive cancellation loss of accuracy that occurs with nearly equal b values. All of these algorithms were implemented in Fortran double precision. The average time for a TVN probability computation using a Fortran 77 (double precision) implementation with an 800 MHZ Pentium III processor was $O(10^{-5})$ seconds for the Plackett formula and Gauss-Hermite algorithm implementations; the Φ^{-1} transform algorithm implementation usually took about 10 times longer.

Some tests were also completed to investigate the possibility of fixed rule double precision TVN algorithms. Some 48-point Gauss rule results are also provided in Table 1. There were significant decreases in the maximum errors, but the results suggest that a reliable double precision algorithm might require a Gauss rule with several hundred points. The use of an adaptive integration was also investigated. A simple globally adaptive algorithm

Table 1: Maximum Errors for TVN Methods for Grid of θ's

	ϵ	Φ^{-1} Transform	Gauss-Hermite	Singular Plackett	Drezner-Plackett
6-Point Rule	0	$1 \cdot 10^{-3}$	$3 \cdot 10^{-3}$	$6 \cdot 10^{-5}$	$1 \cdot 10^{-5}$
12-Point Rule	0	$1 \cdot 10^{-4}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-5}$	$2 \cdot 10^{-7}$
24-Point Rule	0	$2 \cdot 10^{-5}$	$5 \cdot 10^{-4}$	$2 \cdot 10^{-6}$	$8 \cdot 10^{-9}$
24-Point Rule	.01	$2 \cdot 10^{-5}$	$3 \cdot 10^{-4}$	$2 \cdot 10^{-6}$	$8 \cdot 10^{-7}$
48-Point Rule	0	$7 \cdot 10^{-6}$	—	$8 \cdot 10^{-8}$	$6 \cdot 10^{-11}$
48-Point Rule	.01	$2 \cdot 10^{-6}$	—	$9 \cdot 10^{-8}$	$4 \cdot 10^{-10}$
Adaptive	0	$4 \cdot 10^{-9}$	—	$1 \cdot 10^{-11}$	$3 \cdot 10^{-14}$
Adaptive	.01	$4 \cdot 10^{-9}$	—	$4 \cdot 10^{-13}$	$1 \cdot 10^{-13}$
Adaptive Times	0	$2 \cdot 10^{-3}$ s	—	$7 \cdot 10^{-5}$ s	$8 \cdot 10^{-5}$ s

similar to the one used in many of the algorithms for QUADPACK (Piessens, deDoncker, Uberhuber and Kahaner, 1983) was selected. This type of algorithm uses a fixed local integration rule that also provides an error estimate. The rule is first applied to the integrand over the whole integration interval. If the error estimate is larger than the requested accuracy, then the interval, with associated error and integral estimates, is used to initialize a list. The algorithm then proceeds in stages. At each stage an interval with largest error is removed from the list and divided in half. The local integration rule is used on each half of the selected interval and results from the two halves are added to the list. The algorithm terminates when the sum of the local error estimates is less than the requested accuracy. The sum of the local integral estimates is returned as the result. After some experimentation, a 23-point Kronrod rule (see Davis and Rabinowitz, 1984, pp. 106–109) was selected for the local integration rule. This degree thirty-five rule includes an imbedded 11-point Gauss-Legendre rule. The difference between results from the two rules is used to provide a local error estimate. This adaptive algorithm was used with requested accuracy level set at 10^{-14}. The results are given in the last three rows of Table 1. The algorithms based on equations (10), (13) and (14) all had difficulty reliably achieving the requested accuracy level, with the Φ^{-1} transform method providing the worst performance. These difficulties are probably due to the sensitivity to subtractive cancellations for all three of the methods. The error averages for the three methods were $1 \cdot 10^{-12}$, $3 \cdot 10^{-15}$ $3 \cdot 10^{-17}$, respectively, for the $\epsilon = 0$ test. Overall, the tests suggest that the use of equation (14) with an adaptive algorithm can provide double precision results for most TVN problems with times less than 10^{-4} seconds using modern computers. Double precision BVN (BVND) and TVN (part of TVTL) Fortran implementations, with supporting functions, are available from the author's website (www.math.wsu.edu/faculty/genz/homepage, in TVPACK)

4 Bivariate t Probabilities

4.1 The Standard BVT Problem

The standard bivariate t distribution function is defined by

$$T_\nu(\mathbf{b}, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} (1 + \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{(1-\rho^2)\nu})^{-\frac{\nu+2}{2}} dx_2 dx_1, \tag{16}$$

with $\mathbf{b} = (b_1, b_2)$. An alternate definition (Cornish, 1954) for $T_\nu(\mathbf{b}, \rho)$ is

$$T_\nu(\mathbf{b}, \rho) = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)} \int_0^\infty s^{\nu-1} e^{-\frac{s^2}{2}} \Phi(\frac{s}{\sqrt{\nu}}\mathbf{b}, \rho) ds. \tag{17}$$

The Dunnett and Sobel (1954) BVT algorithm has been carefully implemented by the present author. This algorithm uses finite sums of incomplete beta function values which can easily be computed to accuracies that are at the same level as the underlying accuracy of the implementation (e.g single, double or quadruple precision in Fortran). Motivated by work for the development of TVT algorithms, a new BVT algorithm will be described. The new algorithm uses a generalization of the Plackett formula that is the basis for the BVN algorithms that have already been described. This section will end with a report and discussion of test results for the two algorithms.

4.2 A Generalized Plackett BVT Formula

A bivariate t generalization of Plackett's formula requires $\partial T_\nu(\mathbf{b},\rho)/\partial\rho$. If definition (17) is used for $T_\nu(\mathbf{b},\rho)$, and Plackett's formula (2) is used, then

$$\frac{\partial T_\nu(\mathbf{b},\rho)}{\partial\rho} = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)}\int_0^\infty s^{\nu-1}e^{-\frac{s^2}{2}}\frac{e^{-\frac{s^2 f_3(\rho)}{2\nu}}}{2\pi\sqrt{1-\rho^2}}ds.$$

If the exponential terms are combined, and this is followed by the change of variables $r = s(1+\frac{f_3(\rho_{21})}{\nu})^{\frac{1}{2}}$, the result is

$$\frac{\partial T_\nu(\mathbf{b},\rho)}{\partial\rho} = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)}\frac{(1+\frac{f_3(\rho}{\nu})^{-\frac{\nu}{2}}}{2\pi\sqrt{1-\rho^2}}\int_0^\infty r^{\nu-1}e^{\frac{-r^2}{2}}dr.$$

The integral value is $\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}-1}$, so after expanding $f_3(\rho)$, the bivariate t generalization of Plackett's formula is given by

$$\frac{\partial T_\nu(\mathbf{b},\rho)}{\partial\rho} = \frac{(1+\frac{b_1^2+b_2^2-2\rho b_1 b_2}{\nu(1-\rho^2)})^{-\frac{\nu}{2}}}{2\pi\sqrt{1-\rho^2}}. \tag{18}$$

The new formula (18) can be integrated to produce new BVT formulas

$$T_\nu(\mathbf{b},\rho) = T_\nu(\mathbf{b},0) + \frac{1}{2\pi}\int_0^\rho \frac{(1+\frac{b_1^2+b_2^2-2rb_1b_2}{\nu(1-r^2}})^{-\frac{\nu}{2}}}{\sqrt{1-r^2}}dr, \tag{19}$$

and

$$T_\nu(\mathbf{b},\rho) = T_\nu(\mathbf{b},s) + \frac{1}{2\pi}\int_s^\rho \frac{(1+\frac{b_1^2+b_2^2-2rb_1b_2}{\nu(1-r^2)})^{-\frac{\nu}{2}}}{\sqrt{1-r^2}}dr, \tag{20}$$

where, $s = sign(\rho)$ and

$$T_\nu(\mathbf{b},s) = \begin{cases} T_\nu(min(b_1,b_2)) & \text{if s = 1} \\ max(0, T_\nu(b_1) - T_\nu(-b_2)) & \text{if s = -1} \end{cases},$$

with $T_\nu(b)$ defined as the standard univariate Student's t distribution. In contrast to the normal case, there is no easy computation of $T_\nu(\mathbf{b},0)$ that uses a product of univariate t distribution values. Therefore, a numerical implementation of an algorithm based on equation (20) was developed. After the change of variables $r = sin(\theta)$

$$T_\nu(\mathbf{b},\rho) = T_\nu(\mathbf{b},s) + \frac{1}{2\pi}\int_{s\pi/2}^{\sin^{-1}(\rho)}\left(1+\frac{b_1^2+b_2^2-2\sin(\theta)b_1b_2}{\nu\cos(\theta)^2}\right)^{-\frac{\nu}{2}}d\theta, \tag{21}$$

numerical integration can be used to approximate the integral.

4.3 BVT Algorithm Tests

Double precision implementations, using equation (21) with several Gauss-Legendre rules, and with the adaptive algorithm described in the previous section (absolute error tolerance set at 10^{-14}), were tested. The test used $b_1 = -5, -4.75, \ldots, 5$, $b_2 = b_1, b_1 + .25, \ldots, 5$, $\rho = -n/(n+1), (-n+4)/(n+1), \ldots, n/(n+1)$ with $n = 64$, and $\nu = 1, 2, \ldots, 25$, with results given in Table 2, which include average computation time in seconds for the different integration methods. Additional tests with $n = 128, 256, 512$, to check for sensitivity of the algorithms to nearly singular problems, did not produce significantly different results.

A quadruple precision implementation of an algorithm based on the Dunnett and Sobel (1954) paper was used for an accurate comparison. A double precision implementation of the Dunnett and Sobel (1954) paper algorithm reliably achieves double precision results and typically takes about $3 \cdot 10^{-6}$ seconds, approximately one tenth of the time taken by the adaptive algorithm using equation (21). This time difference and results in Table 2 do not support the use of methods that use equation (21) with numerical integration for the efficient computation of BVT probabilities.

Table 2: Generalized Plackett Formula BVT Method Results

	ϵ	Maximum Error	Average Time
6-Point Rule	0	$1 \cdot 10^{-3}$	$2 \cdot 10^{-6}$ s
12-Point Rule	0	$1 \cdot 10^{-4}$	$6 \cdot 10^{-6}$ s
24-Point Rule	0	$1 \cdot 10^{-6}$	$1 \cdot 10^{-5}$ s
24-Point Rule	.01	$2 \cdot 10^{-4}$	$1 \cdot 10^{-5}$ s
48-Point Rule	0	$6 \cdot 10^{-11}$	$2 \cdot 10^{-5}$ s
48-Point Rule	.01	$1 \cdot 10^{-5}$	$2 \cdot 10^{-5}$ s
Adaptive	0	$3 \cdot 10^{-16}$	$3 \cdot 10^{-5}$ s
Adaptive	.01	$6 \cdot 10^{-16}$	$3 \cdot 10^{-5}$ s

5 Trivariate t Probabilities

5.1 The Standard TVT Problem

The standard trivariate t distribution function is defined by

$$T_\nu(\mathbf{b}, R) = \frac{\Gamma((\nu+3)/2)}{\Gamma(\nu/2)\sqrt{(\nu\pi)^3|R|}} \int_{-\infty}^{b_1}\int_{-\infty}^{b_2}\int_{-\infty}^{b_3} (1 + \frac{\mathbf{x}^T R^{-1}\mathbf{x}}{\nu})^{-\frac{\nu+3}{2}} dx_3 dx_2 dx_1, \tag{22}$$

where R is a correlation matrix. An alternate definition (Cornish, 1954) is

$$T_\nu(\mathbf{b}, R) = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)} \int_0^\infty s^{\nu-1} e^{-\frac{s^2}{2}} \Phi(\frac{s}{\sqrt{\nu}}\mathbf{b}, R) ds. \tag{23}$$

In order to develop an efficient TVT algorithm, some experimentation was initially done with equation (23) using algorithms that combine a numerical integration method for the outer integral with an efficient method for the inner TVN integrals. Some of the results of this experimentation will be discussed later in this section, but these methods, which use the four-dimensional integral in equation (23), do not appear to be as efficient as methods that use a generalization of Plackett's formula, and will not be considered in any detail here .

5.2 A Generalized Plackett TVT Formula

A generalization of Plackett's formula requires $\partial T_\nu(\mathbf{b}, R)/\partial\rho_{21}$. If definition (23) is used for $T_\nu(\mathbf{b}, R)$, then

$$\frac{\partial T_\nu(\mathbf{b}, R)}{\partial \rho_{21}} = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)} \int_0^\infty s^{\nu-1} e^{-\frac{s^2}{2}} \frac{\partial \Phi(\frac{s}{\sqrt{\nu}}\mathbf{b}, R)}{\partial \rho_{21}} ds,$$

so Plackett's TVN formula (12) can be applied, giving

$$\frac{\partial T_\nu(\mathbf{b}, R)}{\partial \rho_{21}} = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)} \int_0^\infty s^{\nu-1} e^{-\frac{s^2}{2}} \frac{e^{-\frac{s^2 f_3(\rho_{21})}{2\nu}}}{(2\pi)^{\frac{3}{2}}\sqrt{1-\rho_{21}^2}} \int_{-\infty}^{\frac{s}{\nu}u_3(\rho_{21})} e^{-\frac{x^2}{2}} dx ds.$$

If the inner integral is transformed using $x = \frac{s}{\sqrt{\nu}}y$, the exponential terms are combined, a second change of variables $r = s(1 + \frac{f_3(\rho_{21})+y^2}{\nu})^{\frac{1}{2}}$ is used, and the order of integration is reversed, the result is

$$\frac{\partial T_\nu(\mathbf{b}, R)}{\partial \rho_{21}} = \frac{2^{1-\nu/2}}{\sqrt{\nu}\Gamma(\nu/2)} \int_{-\infty}^{u_3(\rho_{21})} \frac{(1 + \frac{f_3(\rho_{21})+y^2}{\nu})^{-\frac{\nu+1}{2}}}{(2\pi)^{\frac{3}{2}}\sqrt{1-\rho_{21}^2}} \int_0^\infty r^\nu e^{-\frac{r^2}{2}} dr dy.$$

Next, the factorization $(1 + \frac{f_3(\rho)+y^2}{\nu}) = (1 + \frac{f_3(\rho)}{\nu})(1 + \frac{y^2}{\nu+f_3(\rho)})$ is used, along with the inner integral value $\Gamma(\frac{\nu+1}{2})2^{\frac{\nu+1}{2}-1}$, and some algebra, to obtain

$$\frac{\partial T_\nu(\mathbf{b}, R)}{\partial \rho_{21}} = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \frac{(1 + \frac{f_3(\rho_{21})}{\nu})^{-\frac{\nu+1}{2}}}{2\pi\sqrt{1-\rho_{21}^2}} \int_{-\infty}^{u_3(\rho_{21})} (1 + \frac{y^2}{\nu + f_3(\rho_{21})})^{-\frac{\nu+1}{2}} dy.$$

The final change of variables, $z = y/(1+\frac{f_3(\rho_{21})}{\nu})^{\frac{1}{2}}$, and some more algebra, produces the new trivariate t formula

$$\frac{\partial T_\nu(\mathbf{b},R)}{\partial \rho_{21}} = \frac{(1+\frac{f_3(\rho_{12})}{\nu})^{-\frac{\nu}{2}}}{2\pi\sqrt{1-\rho_{21}^2}} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \int_{-\infty}^{u_3(\rho_{21})/(1+\frac{f_3(\rho_{21})}{\nu})^{\frac{1}{2}}} (1+\frac{z^2}{\nu})^{-\frac{\nu+1}{2}} dz.$$

This can be written in a form very similar to the form for equation (12) as

$$\frac{\partial T_\nu(\mathbf{b},R)}{\partial \rho_{21}} = \frac{(1+\frac{f_3(\rho_{21})}{\nu})^{-\frac{\nu}{2}}}{2\pi\sqrt{1-\rho_{21}^2}} T_\nu\big(\frac{u_3(\rho_{21})}{(1+\frac{f_3(\rho_{21})}{\nu})^{\frac{1}{2}}}\big). \tag{24}$$

Appropriate permutations of the b's and the ρ's can be used to provide similar formulas for $\partial T_\nu(\mathbf{b},R)/\partial\rho_{31}$ and $\partial T_\nu(\mathbf{b},R)/\partial\rho_{32}$.

5.3 Generalized Plackett TVT Algorithms

The formula (24) can be integrated to produce new TVT formulas (analogous to (13-14))

$$T_\nu(\mathbf{b},R) = T_\nu(\mathbf{b},R^*) + \frac{1}{2\pi}\int_{\rho_{21}^*}^{\rho_{21}} \frac{(1+\frac{f_3(r)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-r^2}} T_\nu\big(\frac{u_3(r)}{(1+\frac{f_3(r)}{\nu})^{\frac{1}{2}}}\big)dr \tag{25}$$

and

$$\begin{aligned} T_\nu(\mathbf{b},R) = T_\nu(\mathbf{b},R') \quad + \quad & \frac{1}{2\pi}\int_0^1 \Bigg(\rho_{21}\frac{(1+\frac{f_3(\rho_{21}t)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-\rho_{21}^2t^2}} T_\nu\big(\frac{\hat{u}_3(t)}{(1+\frac{f_3(\rho_{21}t)}{\nu})^{\frac{1}{2}}}\big) \\ & + \rho_{31}\frac{(1+\frac{f_2(\rho_{31}t)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-\rho_{31}^2t^2}} T_\nu\big(\frac{\hat{u}_2(t)}{(1+\frac{f_2(\rho_{31}t)}{\nu})^{\frac{1}{2}}}\big) \Bigg) dt. \end{aligned} \tag{26}$$

The most efficient method for the TVN case used equation (14), so this suggests that equation (26) could be used for an efficient TVT method. Unfortunately, there is no easy computation of $T_\nu(\mathbf{b},R')$ that uses univariate and bivariate t distribution values. However, a combination of equations (25) and (26) can be used for a practical method. First define

$$R^{**} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & s & 1 \end{bmatrix},$$

with $s = sign(\rho_{32})$. If equation (25) is used, with variables 1 and 3 interchanged, to compute $T_\nu(\mathbf{b},R')$ from $T_\nu(\mathbf{b},R^{**})$, then

$$T_\nu(\mathbf{b},R') = T_\nu(\mathbf{b},R^{**}) + \frac{1}{2\pi}\int_s^{\rho_{32}} \frac{(1+\frac{f_1(r)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-r^2}} T_\nu\big(\frac{b_1}{(1+\frac{f_1(r)}{\nu})^{\frac{1}{2}}}\big)dr,$$

where $f_1(r) = \frac{b_2^2+b_3^2-2rb_2b_3}{(1-r^2)}$. The simplified b_1 term in the integrand's T_ν argument numerator occurs because the zeros in R^{**} result in $u_1(r) = b_1$. The equation for the combined method is

$$\begin{aligned} T_\nu(\mathbf{b},R) = T_\nu(\mathbf{b},R^{**}) \quad + \quad & \frac{1}{2\pi}\int_s^{\rho_{32}} \frac{(1+\frac{f_1(r)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-r^2}} T_\nu\big(\frac{b_1}{(1+\frac{f_1(r)}{\nu})^{\frac{1}{2}}}\big)dr \\ + \quad & \frac{1}{2\pi}\int_0^1 \Bigg(\rho_{21}\frac{(1+\frac{f_3(\rho_{21}t)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-\rho_{21}^2t^2}} T_\nu\big(\frac{\hat{u}_3(t)}{(1+\frac{f_3(\rho_{21}t)}{\nu})^{\frac{1}{2}}}\big) \\ & + \rho_{31}\frac{(1+\frac{f_2(\rho_{31}t)}{\nu})^{-\frac{\nu}{2}}}{\sqrt{1-\rho_{31}^2t^2}} T_\nu\big(\frac{\hat{u}_2(t)}{(1+\frac{f_2(\rho_{31}t)}{\nu})^{\frac{1}{2}}}\big) \Bigg) dt. \end{aligned} \tag{27}$$

The singular $T_\nu(\mathbf{b},R^{**})$ can be efficiently computed with BVT values using

$$T_\nu(\mathbf{b},R^{**}) = \begin{cases} T_\nu((b_1, min(b_2,b_3)),0) & \text{if s = 1} \\ max(0, T_\nu((b_1,b_2),0) - T_\nu((b_1,-b_3),0)) & \text{if s = -1} \end{cases},$$

where $T_\nu((b_1,b_2),\rho)$ is the standard bivariate t distribution function.

Table 3: Maximum (Average) Errors for TVT Methods for Grid of θ's, $\nu = 1$

	ϵ	T_ν^{-1} Transformed	Singular Generalized	Combined Generalized
6-Point Rule	0	$1 \cdot 10^{-3}$ $(1 \cdot 10^{-5})$	$6 \cdot 10^{-5}$ $(7 \cdot 10^{-7})$	$4 \cdot 10^{-5}$ $(7 \cdot 10^{-7})$
12-Point Rule	0	$2 \cdot 10^{-4}$ $(1 \cdot 10^{-6})$	$2 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$1 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$
24-Point Rule	0	$3 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$5 \cdot 10^{-6}$ $(2 \cdot 10^{-8})$	$2 \cdot 10^{-6}$ $(1 \cdot 10^{-8})$
24-Point Rule	.01	$3 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$2 \cdot 10^{-5}$ $(2 \cdot 10^{-7})$	$2 \cdot 10^{-5}$ $(3 \cdot 10^{-7})$
48-Point Rule	0	$1 \cdot 10^{-5}$ $(3 \cdot 10^{-8})$	$6 \cdot 10^{-7}$ $(3 \cdot 10^{-9})$	$1 \cdot 10^{-7}$ $(1 \cdot 10^{-9})$
48-Point Rule	.01	$7 \cdot 10^{-6}$ $(2 \cdot 10^{-8})$	$1 \cdot 10^{-6}$ $(7 \cdot 10^{-9})$	$4 \cdot 10^{-6}$ $(7 \cdot 10^{-9})$
Adaptive	0	$4 \cdot 10^{-15}$ $(2 \cdot 10^{-17})$	$2 \cdot 10^{-11}$ $(7 \cdot 10^{-16})$	$8 \cdot 10^{-14}$ $(3 \cdot 10^{-17})$
Adaptive	.01	$2 \cdot 10^{-14}$ $(2 \cdot 10^{-17})$	$1 \cdot 10^{-12}$ $(6 \cdot 10^{-16})$	$2 \cdot 10^{-13}$ $(3 \cdot 10^{-17})$
Adaptive Times	0	$2 \cdot 10^{-4}$ s	$9 \cdot 10^{-4}$ s	$3 \cdot 10^{-4}$ s

5.4 TVT Algorithm Tests

A method similar to the TVN method based on equation (10) is also considered here. In this case the TVT distribution can be written in the form

$$T_\nu(\mathbf{b}, R) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}} \int_{-\infty}^{b_1} (1 + x^2/\nu)^{-\frac{\nu+1}{2}} G(x) dx,$$

where

$$G(x) = T_\nu\left(\left(\frac{(\nu+1)^{\frac{1}{2}}(b_2 - \rho_{21}x)}{((\nu+x^2)(1-\rho_{21}^2))^{\frac{1}{2}}}, \frac{(\nu+1)^{\frac{1}{2}}(b_3 - \rho_{31}x)}{((\nu+x^2)(1-\rho_{21}^3))^{\frac{1}{2}}} \right), \frac{\rho_{32} - \rho_{31}\rho_{21}}{((1-\rho_{21}^2)(1-\rho_{31}^2))^{\frac{1}{2}}} \right).$$

The transformation $x = T_\nu^{-1}(t)$ produces the formula

$$T_\nu(\mathbf{b}, R) = \int_0^{T_\nu(b_1)} G(T_\nu^{-1}(t)) dt, \tag{28}$$

which can be used with numerical integration to compute TVT probabilities.

Algorithms that combine numerical integration with equations (25), (27) and (28) were implemented and tested in a manner similar to that used for the TVN algorithms where the correlation matrices depend on the C (15) matrices with $\theta_1 = 1/258, 17/258, ..., 257/258$, $\theta_2 = 1/258, 17/258, ..., 257/258$, $\theta_3 = 1/258, 17/258, ..., 257/258$, and with limits $b_1 = -5, -4, ..., 5$, $b_2 = -5, -4, ..., 5$, $b_3 = b_2, b_2 + 1, ..., 5$. Some tests (indicated with $\epsilon > 0$) were also completed with nearly equal b values. The adaptive integration algorithm used for the three methods had an absolute error tolerance level set at 10^{-14}. A quadruple precision implementation of the adaptive integration algorithm applied to equation (27) with absolute error tolerance level set at 10^{-18} was used to provide high accuracy TVT values for comparisons. This algorithm included a quadruple precision Dunnett and Sobel (1954) algorithm implementation for special case BVT probabilities.

Tables 3, 4 and 5 provide results using Gauss rule and adaptive integration for methods using equations (28), (25) and (27) (respectively referred to as the T_ν^{-1} Transformed, Singular Generalized and Combined Generalized methods) for $\nu = 1, 5, 25$. Average errors are given in parenthesis for each table entry. The last line of each Table provides average times in seconds for the adaptive algorithms using an 800 MHZ Pentium III computer. These times are significantly smaller than typical times for general-purpose algorithms developed for multivariate normal and multivariate t probability computations (see Genz and Bretz, 2002, and Genz, 1993), where high accuracy computations are often infeasible. Some testing was also done using adaptive integration for a method based on equation (23). The outer infinite integration interval was transformed to the adaptive integration interval $[0, 1]$ using the $\chi_\nu^{-1}(s)$ function, and a Drezner-Plackett method with a fixed integration rule was used for the inner TVN integral. This method cannot achieve accuracy comparable to the other TVT methods discussed in this section and average computation times were more than one hundred times larger than average computation times for the generalized Plackett formula TVT method implementations.

The combined generalized Plackett formula method (equation (27)) provides the highest level of overall accuracy. An algorithm based on this method with only a 6-point Gauss rule can provide single precision accuracy for most TVT problems. An adaptive algorithm (with Fortran implementation TVTL in TVPACK, available from the author's website) can provide double precision accuracy for most TVT problems. The algorithm that uses adaptive integration with the singular generalized Plackett formula is often faster than the combined generalized Plackett formula algorithm, and this difference is more significant for the larger ν values, but this algorithm (like the related TVN algorithm) appears to be more sensitive to loss of accuracy from rounding errors.

Table 4: Maximum (Average) Errors for TVT Methods for Grid of θ's, $\nu = 5$

	ϵ	T_ν^{-1} Transformed	Singular Generalized	Combined Generalized
6-Point Rule	0	$1 \cdot 10^{-3}$ $(8 \cdot 10^{-6})$	$6 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$2 \cdot 10^{-5}$ $(6 \cdot 10^{-8})$
12-Point Rule	0	$1 \cdot 10^{-4}$ $(8 \cdot 10^{-7})$	$2 \cdot 10^{-5}$ $(2 \cdot 10^{-8})$	$1 \cdot 10^{-6}$ $(3 \cdot 10^{-9})$
24-Point Rule	0	$3 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$8 \cdot 10^{-7}$ $(1 \cdot 10^{-9})$	$1 \cdot 10^{-7}$ $(2 \cdot 10^{-10})$
24-Point Rule	.01	$3 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$8 \cdot 10^{-5}$ $(2 \cdot 10^{-7})$	$8 \cdot 10^{-5}$ $(4 \cdot 10^{-7})$
48-Point Rule	0	$1 \cdot 10^{-5}$ $(3 \cdot 10^{-8})$	$2 \cdot 10^{-7}$ $(2 \cdot 10^{-10})$	$3 \cdot 10^{-9}$ $(4 \cdot 10^{-12})$
48-Point Rule	.01	$3 \cdot 10^{-6}$ $(2 \cdot 10^{-8})$	$1 \cdot 10^{-6}$ $(4 \cdot 10^{-9})$	$1 \cdot 10^{-6}$ $(7 \cdot 10^{-9})$
Adaptive	0	$5 \cdot 10^{-7}$ $(2 \cdot 10^{-12})$	$1 \cdot 10^{-11}$ $(7 \cdot 10^{-16})$	$1 \cdot 10^{-13}$ $(2 \cdot 10^{-17})$
Adaptive	.01	$5 \cdot 10^{-7}$ $(4 \cdot 10^{-12})$	$1 \cdot 10^{-12}$ $(5 \cdot 10^{-16})$	$2 \cdot 10^{-13}$ $(3 \cdot 10^{-17})$
Adaptive Times	0	$5 \cdot 10^{-3}$ s	$2 \cdot 10^{-4}$ s	$4 \cdot 10^{-4}$ s

Table 5: Maximum (Average) Errors for TVT Methods for Grid of θ's, $\nu = 25$

	ϵ	T_ν^{-1} Transformed	Singular Generalized	Combined Generalized
6-Point Rule	0	$1 \cdot 10^{-3}$ $(6 \cdot 10^{-6})$	$6 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$	$1 \cdot 10^{-5}$ $(1 \cdot 10^{-7})$
12-Point Rule	0	$1 \cdot 10^{-4}$ $(5 \cdot 10^{-7})$	$2 \cdot 10^{-5}$ $(2 \cdot 10^{-8})$	$4 \cdot 10^{-7}$ $(1 \cdot 10^{-9})$
24-Point Rule	0	$2 \cdot 10^{-5}$ $(6 \cdot 10^{-8})$	$1 \cdot 10^{-6}$ $(1 \cdot 10^{-9})$	$9 \cdot 10^{-9}$ $(2 \cdot 10^{-11})$
24-Point Rule	.01	$2 \cdot 10^{-5}$ $(5 \cdot 10^{-8})$	$1 \cdot 10^{-4}$ $(3 \cdot 10^{-7})$	$1 \cdot 10^{-4}$ $(7 \cdot 10^{-7})$
48-Point Rule	0	$5 \cdot 10^{-6}$ $(6 \cdot 10^{-6})$	$3 \cdot 10^{-8}$ $(5 \cdot 10^{-11})$	$4 \cdot 10^{-10}$ $(5 \cdot 10^{-13})$
48-Point Rule	.01	$3 \cdot 10^{-6}$ $(1 \cdot 10^{-8})$	$6 \cdot 10^{-6}$ $(2 \cdot 10^{-8})$	$6 \cdot 10^{-6}$ $(4 \cdot 10^{-8})$
Adaptive	0	$1 \cdot 10^{-6}$ $(6 \cdot 10^{-12})$	$1 \cdot 10^{-11}$ $(7 \cdot 10^{-16})$	$1 \cdot 10^{-13}$ $(5 \cdot 10^{-17})$
Adaptive	.01	$1 \cdot 10^{-6}$ $(1 \cdot 10^{-11})$	$1 \cdot 10^{-12}$ $(6 \cdot 10^{-16})$	$2 \cdot 10^{-13}$ $(5 \cdot 10^{-17})$
Adaptive Times	0	$6 \cdot 10^{-3}$ s	$2 \cdot 10^{-4}$ s	$6 \cdot 10^{-4}$ s

6 Conclusions

Algorithms for accurate and efficient computations of BVN, TVN, BVT and TVT probabilities were considered. Generalizations of Plackett's Normal formulas were derived for the bivariate t and trivariate t cases. An implementation of an algorithm using the generalized bivariate t formula was significantly slower than an implementation of the Dunnett and Sobel (1954) algorithm. The Dunnett and Sobel algorithm continues to be the most efficient algorithm for BVT computations. Although Plackett formula algorithms for TVN and TVT probabilities are susceptible to moderate loss of accuracy for some types of problems, implementations of algorithms that use adaptive integration can efficiently produce double precision results for most problems, with times that are less than $O(10^{-4})$ seconds for Fortran implementations running on modern computer workstations. Fortran software for all of the algorithms discussed in this paper is available from the author's website.

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