CHEBYSHEV INTERPOLATION

Background

- Interpolation error is:

\[ f(x) - P_n(x) = \prod_{i=0}^{n} (x - x_i) \frac{f^{(n+1)}(\xi(x))}{(n + 1)!}, \]

- Goal: try to reduce interpolation error by choosing \(x_i\)'s to minimize \(\|w(x)\| = \max_{x \in [a,b]} |\prod_{i=0}^{n} (x - x_i)|\).

Chebyshev polynomials

- The degree \(n\) Chebyshev polynomial is

\[ T_n(x) = \cos(n \cos^{-1}(x)). \]

Notice: \(T_0(x) = 1, T_1(x) = x, T_2(x) = \cos(2 \cos^{-1}(x)) = 2x^2 - 1.\)

- Chebyshev polynomial recursion relation:
  starting with \(T_0(x) = 1, T_1(x) = x,\)

\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \]

Examples: \(T_3(x), T_4(x)\)
• Chebyshev Polynomial Graphs
• Chebyshev polynomial properties:
  a) $T_n(x) = 2^{n-1}x^n + \text{lower degree terms.}$

  b) $T_n(1) = 1, T_n(-1) = (-1)^n.$

  c) $T_n(x)$ has zeros at $x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right), i = 1, \ldots, n.$

  \[ T_n(x) = 2^{n-1} \prod_{i=0}^{n} (x - x_i). \]

  d) $T_n(x) = (-1)^i$ at $u_i = \cos\left(\frac{i\pi}{n}\right), i = 0, \ldots, n.$
CHEBYSHEV INTERPOLATION CONT.

• Chebyshev Polynomial Theorem: $2^{1-n}T_n(x)$ is a monic degree $n$ polynomial $P_n(x)$ (monic means $= x^n +$ lower degree) which minimizes $\max_{x \in [-1,1]} |P_n(x)|$.

Proof: notice $\max_{x \in [-1,1]} |2^{1-n}T_n(x)| = 2^{1-n}$.

Now assume $\max_{x \in [-1,1]} |P_n(x)| < 2^{1-n}$, and check $E(x) = 2^{1-n}T_n(x) - P_n(x)$ at $u_i$’s.
Chebyshev Interpolation

- If the interpolation interval is $[a, b] = [-1, 1]$, use $x_i = \cos\left(\frac{(2i-1)\pi}{2(n+1)}\right)$, $i = 1, \ldots, n + 1$; then

$$|f(x) - P_n(x)| = |2^{-n}T_{n+1}(x)\frac{f^{(n+1)}(\xi(x))}{(n+1)!}| \leq \frac{B_{n+1}}{2^n(n+1)!},$$

with $B_{n+1} = \max_{x \in [-1,1]} |f^{(n+1)}(x)|$.

- Change of interval: if $[a, b] \neq [-1, 1]$, use

$$x_i = \frac{b + a}{2} + \frac{b - a}{2} \cos\left(\frac{(2i - 1)\pi}{2(n+1)}\right),$$

for $i = 1, \ldots, n + 1$; then

$$|f(x) - P_n(x)| \leq \left(\frac{b - a}{2}\right)^{n+1} \frac{B_{n+1}}{2^n(n+1)!},$$

with $B_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)|$.  


Example: $\sin(x)$ for $x \in [0, \pi/2]$.
Interpolation points are $x_i = \frac{\pi}{4} + \frac{\pi}{4} \cos \left( \frac{(2i-1)\pi}{2(n+1)} \right)$.
Error bound for $n = 7$ is
$$| \sin(x) - P_7(x) | \leq \left( \frac{\pi}{4} \right)^8 \frac{1}{2^78!} \approx 2.8 \times 10^{-8}.$$ 
Actual $P_7(x)$ has max $| \sin(x) - P_7(x) | \approx 2.1 \times 10^{-8}$

n = 7; x=pi*(1+cos(pi*[1:2:2*n+1]/(2*n+2)))/4; f = sin(x); xt = pi*[0:128]/256; fv = lgrang( x,f,xt ); plot( xt, fv-sin(xt) );
Note: max $| \sin(x) - P_6(x) | \approx 4.3 \times 10^{-7}$, so degree 6 provides single precision accurate poly for $\sin(x)$.
Taylor series needs $n = 11$ for $\approx 4.5 \times 10^{-7}$ error.
Double precision for Chebyshev needs $n = 13$. 
Runge Phenomenon Example

- consider \( f(x) = \frac{1}{1 + 12x^2} \)
  interpolated for \( x \in [-1, 1] \) with Chebyshev \( x_i \)'s.

Maximum errors for \( x \in [-1, 1] \):

- .2 (for \( P_7(x) \)), .06 (\( P_{11}(x) \)), .02 (\( P_{15}(x) \)), .007 (\( P_{19}(x) \)).
CHEBYSHEV INTERP. CONT.

Matlab for Lagrange interpolation function using
efficient evaluation in the form

\[ P_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \]

\[ = \prod_{i=0}^{n} (x - x_i) \sum_{i=0}^{n} \frac{f(x_i)}{w_i(x - x_i)} \]

\[ = \sum_{i=0}^{n} \frac{f(x_i)}{w_i(x - x_i)} / \sum_{i=0}^{n} \frac{1}{w_i(x - x_i)} \]

with \( w_i = \prod_{j=0, j \neq i}^{n} (x_i - x_j) \).

function p = lgrang( x, f, xt )
    n = length(x); m = length(xt);
    % Evaluates Lagrange interpolating poly
    % for the data vectors x and f,
    % at the points xt(1), ..., xt(m),
    % returning results in p(1), ..., p(m)
    for i = 1 : n % % determine weights
        w(i) = prod( x(i) - x([1:i-1 i+1:n]));
    end
    for l = 1 : m % % determine poly values at xt’s
        wt = w.*( xt(l) - x );
        p(l) = sum(f./wt)/sum(1./wt);
    end