Course Purposes

- To develop understanding of mathematical background for widely used algorithms for solving numerical problems with computers;
- To develop understanding of practical difficulties associated with implementing and using numerical algorithms.

What is Numerical Analysis?

The construction and analysis of methods for the approximate solution of mathematically posed problems that have no simple analytic (formula) solutions.

Some examples:

- \( \int_0^x \cos(t)dt = \sin(x) \), but \( \int_0^2 \cos(t^3)dt =?\)
- \( x^2 + 3x + 1 = 0 \) \( \Rightarrow x = \frac{-3+\sqrt{5}}{2} \), but if \( 3x^6 + 2x^3 + x^2 - 5 = 0 \), \( x =?\)
- For matrix \( A \), vector \( b \), \( Ax = b \) \( \Rightarrow x = A^{-1}b \), but how to find \( x \) (and \( A^{-1} \)) efficiently?
Main Subject Areas of Numerical Analysis

1. Background: algebra and analysis, computer arithmetic.

2. Solution of Nonlinear Equations:
   given \( f(x) \), if \( f(x) = 0 \), find \( x \).

3. Solution of Linear Systems, and Eigenproblems:
   solve \( Ax = b \), solve \( Ax = \lambda x \).

4. Approximation: interpolation, numerical integration, data fitting and function representation, e.g. how to efficiently compute \( \sin(x) \),
   \[
   \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
   \]

5. Solution of ODE’s and PDE’s: e.g.
   if \( y' + 5 \cos(x) + y^2 = 0 \), find \( y(x) \)
   if \( y_{xx} + y_{tt} + \cos(y)y = \sin(x) + \tan(t) \), find \( y(x, t) \)

Math 4/548, Cpt S 4/530 Course Coverage:
1, 2, some of 3, 4, and 5; see course schedule.
SOME CALCULUS BACKGROUND

**Basic Definitions:** limits, continuity, derivatives, integrals.

**Basic Theorems:** Rolle’s Theorem, Mean Value Theorem(s), Extreme Value Theorem, Intermediate Value Theorem.
Taylor’s Theorem: given \( f(x), x_0, \)

\[
f(x) \approx P_n(x) = f(x_0) + (x-x_0)f'(x_0) + \ldots + (x-x_0)^n f^{(n)}(x_0)/n!,
\]

with error or remainder

\[
R_n(x) = (x-x_0)^{n+1} f^{(n+1)}(\xi(x))/(n+1)!, \quad \xi(x) \in (x_0, x).
\]

Some examples

- for \( x_0 = 0, \sin(x) = x - x^3/6 + x^5/120, \) error?

- if \( x_0 = 0, f(x) = e^x \cos(x), \)
  find \( P_2(x), R_2(x); \) bound for \( R_2(x). \)
Polynomials and Polynomial Evaluation

- Many numerical analysis methods are based on the use of polynomials:

\[ P(x) = c_1 + c_2x + c_3x^2 + \cdots + c_{d+1}x^d \]

is degree \( d \) polynomial with coefficients \( c_1, c_2, \ldots, c_{d+1} \).

Example \( P(x) = 1 - 2x + 3x^2 + 4x^3 - 5x^4 \).

Need to compute \( P(x) \) for various \( x \); \( P(2) \)?

- Efficient polynomial evaluation.

  Possible methods:
  a) Direct

  b) Powers first

  c) Nested (Horner’s) Method
Arithmetic needed for different methods:
a) Direct

\[ + : d; \star : d(d + 1)/2. \]

b) Powers first

\[ + : d; \star : 2d - 1. \]

c) Nested (Horner’s) Method, Synthetic Division

\[ + : d; \star : d. \]

Algorithm: to find \( P(x_0) \)
1) set \( P = c_{d+1} \)
2) for \( i = d : -1 : 1 \)
   set \( P = c_i + x_0 \star P \)
   end for
3) Output \( P(x_0) = P \).

- Modified algorithm can be used for more general form

\[
P(x) = c_1 + c_2(x - r_1) + c_3(x - r_1)(x - r_2) + \cdots + c_{d+1}(x - r_1) \cdots (x - r_d).
\]
• Horner’s Methods for Derivatives: use

\[ P(x) = P(x_0) + (x - x_0)Q(x), \ P'(x_0) = Q(x_0) \]

with \( Q(x) = b_2 + b_3x + \cdots + b_{d+1}x^{d-1} \)

With \( b_{d+1} = c_{d+1}, \ b_i = c_i + b_{i+1}x_0, \) for \( i = d, \ldots, 2. \)

Algorithm: to find \( P(x_0), \ P'(X_0) \)

1) set \( P = c_{d+1}, \ Q = P \)

2) for \( i = d : -1 : 2 \)
   set \( P = c_i + x_0 \times P, \ Q = P + x_0 \times Q \)
   end for

3) Output \( P(x_0) = c_1 + x_0 \times P, \ P'(x_0) = Q. \)
“Numerical analysis problems do have exact solutions, but the thrill of victory does not wait for their discovery. It is enough to come close. Error is expected. Without it we would be out of business.”

Francis Scheid,
*2000 Solved Problems in Numerical Analysis*.

“In a world in which the price of calculation continues to decrease rapidly, but the price of theorem proving continues to hold steady or increase, elementary economics indicates that we ought to spend a larger and larger fraction of our time on calculation.”

John W. Tukey,