IMPORTANCE SAMPLING

Importance Sampling Background: let $\mathbf{x} = (x_1, \ldots, x_n)$,

$$\theta = E[h(\mathbf{X})] = \int f(\mathbf{x})h(\mathbf{x})d\mathbf{x} \approx \frac{1}{N}\sum_{i=1}^{N} h(\mathbf{X}_i) = \bar{\Theta},$$

if $\mathbf{X}_i \sim F(\mathbf{X})$, and $F(\mathbf{x})$ is cdf for $f(\mathbf{x})$. For many problems, $F(\mathbf{x})$ is difficult to sample from and/or $Var(h)$ is large.

- If a related, easily sampled pdf $g(\mathbf{x})$ is available, you could use

$$\theta = E_g[\frac{f(\mathbf{X})h(\mathbf{X})}{g(\mathbf{X})}] = \int g(\mathbf{x})\frac{f(\mathbf{x})h(\mathbf{x})}{g(\mathbf{x})}d\mathbf{x} \approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(\mathbf{X}_i)h(\mathbf{X}_i)}{g(\mathbf{X}_i)},$$

  with $\mathbf{X}_i \sim G(\mathbf{X})$, for associated cdf $G(\mathbf{X})$.

- **Importance sampling**: if $Var(\frac{h(\mathbf{x})f(\mathbf{x})}{g(\mathbf{x})})$ is small, $g(\mathbf{x})$ samples are concentrated where $h(\mathbf{x})f(\mathbf{x})$ is "important":

IMPORTANCE SAMPLING CONT.

- Importance Sampling Example: $\theta = \int_0^1 e^{x^2} dx$.
  Try $g(x) = e^x$; so $\theta = \int_0^1 \; e^x \; e^{x^2-x} \; dx$, but $g(x)$ is not a an actual pdf.
  To find a correctly scaled $G(x)$ for e^x, note $\int_0^1 e^x dx = e - 1$, so use

$$G(x) = \frac{1}{e-1}\int_0^x e^t dt = (e^x - 1)/(e-1) \qquad = U;$$

  then compute $X_i = \ln(\ 1 \ + \ (e-1)U_i\ )$, so

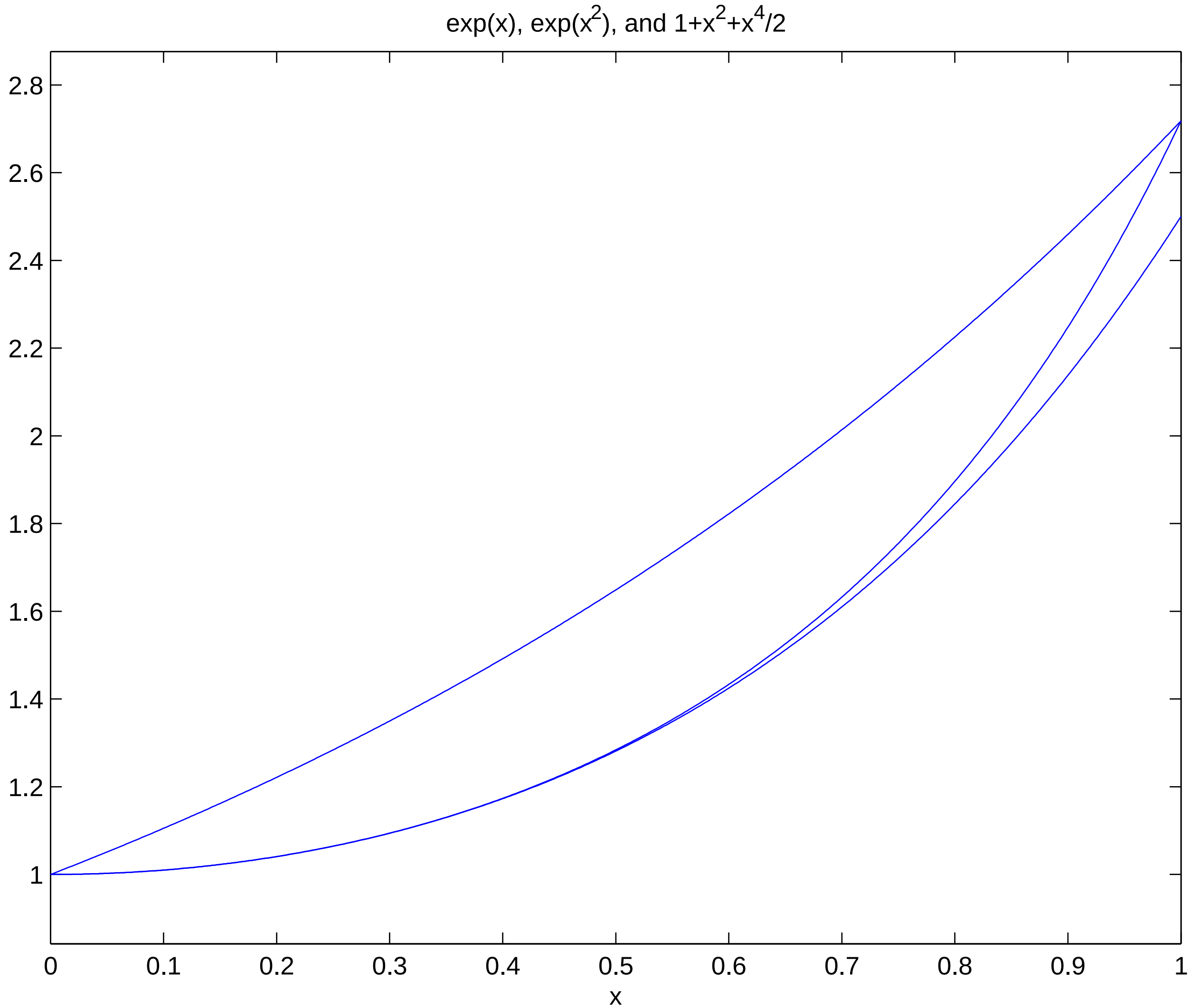
$$\theta = (e-1)\int_0^1 \frac{e^x}{e-1} e^{x^2-x} dx \approx \frac{(e-1)}{N}\sum_{i=1}^N e^{X_i^2 - X_i},$$

```
N = 10000; U = rand(1,N); Y = exp(U.^2);
disp( [mean(Y) 2*std(Y)/sqrt(N)]) % simple MC
       1.4672     0.009463
e = exp(1); X = log(1+(e-1)*U);
T = (e-1)*exp(X.*(X-1));
disp( [mean(T) 2*std(T)/sqrt(N)]) % importance
       1.4628    0.0022348
```

  Error estimate reduction by $\approx 1/4$.

IMPORTANCE SAMPLING CONT.

Graphs of $e^x, e^{x^2}, 1 + x^2 + x^4/2$.



IMPORTANCE SAMPLING CONT.

Alternative: $g(x) = 1 + x^2$, $G(x)$?

$$\theta = \frac{4}{3}\int_0^1 \frac{3(1+x^2)}{4}\frac{e^{x^2}}{1+x^2}\,dx \;\approx\; \frac{4}{3N}\sum_{i=1}^{N}\frac{e^{X_i^2}}{1+X_i^2},$$

with $X_i \sim \frac{3}{4}X + \frac{1}{4}X^3$.

```
N = 10000; U = rand(1,N); I = rand(1,N)<3/4;
X = I.*U + (1-I).*U.^(1/3); T = 4*exp(X.^2)./(3*(1+X.^2));
disp( [mean(T) 2*std(T)/sqrt(N)]) % importance
      1.4627    0.0028178
```

Better $g(x) = 1 + x^2 + x^4/2$, $G(x) = \frac{30}{43}x + \frac{10}{43}x^3 + \frac{3}{43}x^5$;

$$\theta = \frac{43}{30}\int_0^1 \frac{30(1+x^2+x^4/2)}{43}\frac{e^{x^2}}{1+x^2+x^4/2}\,dx \approx \frac{43}{30N}\sum_{i=1}^{N}\frac{e^{X_i^2}}{1+X_i^2+X_i^4/2},$$

with $X_i \sim \frac{30}{43}x + \frac{10}{43}x^3 + \frac{3}{43}x^5$.

```
N = 10000; U = rand(1,N); V = rand(1,N); I = V<30/43; J = V>40/43;
X = I.*U + (1-I).*(1-J).*U.^(1/3)+ J.*U.^(1/5);
T = 43*exp(X.^2)./(30*(1+X.^2+X.^4/2));
disp( [mean(T) 2*std(T)/sqrt(N)]) % importance sampled
      1.4623   0.00072228
```

IMPORTANCE SAMPLING CONT.

- Importance Sampling Example: $\theta = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2}z^3\ e^z\ dz$.

```
N = 100000; Z = randn(1,N); Y = Z.^3.*exp(Z);
disp( [mean(Y) 2*std(Y)/sqrt(N)]) % simple MC
       6.5418      0.33644
```

Try $g(z) = \frac{1}{\sqrt{2\pi}}e^{-(z-1)^2/2} = \frac{1}{\sqrt{2\pi}}e^{-w^2/2}$, with $w = z - 1$; so

$$\theta = e^{1/2}\int_{-\infty}^{\infty} g(z)z^3\ dz = \frac{e^{1/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-w^2/2}(w+1)^3 dw.$$

Simulation uses $\theta \approx \frac{e^{1/2}}{N}\sum_{i=1}^{N}(Z_i+1)^3$, with $Z_i \sim Normal(0,1)$.

```
N = 100000; Z = randn(1,N); Y = exp(1/2)*(Z+1).^3;
disp( [mean(Y) 2*std(Y)/sqrt(N)]) % importance
       6.6338     0.081528
```

Error reduction by $\approx 1/4$. Note

$$\theta = \frac{e^{1/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(3w^2+1)e^{-w^2/2}dw = 4e^{1/2} \approx 6.5948851.$$

IMPORTANCE SAMPLING CONT.

- Higher dimensional problems: often $f(\mathbf{x}) \approx g(\mathbf{x}) = g_1(x_1)g_2(x_2)\cdots g_n(x_n)$, so samples are from a sequence of 1-d samples. 2-d example:

$\theta = \int_0^1\int_0^1 e^{(x_1+x_2)^2} d\mathbf{x}$; if $g(\mathbf{x}) = e^{x_1}e^{x_2}$; $\theta = \int_0^1\int_0^1 e^{x_1+x_2}e^{((x_1+x_2)^2-x_1-x_2}\ d\mathbf{x}$.
After scaling, with $X_{ij} = \ln(1+(e-1)U_{ij})$,

$$\theta = (e-1)^2\int_0^1\int_0^1 \frac{e^{x_1+x_2}}{(e-1)^2}e^{((x_1+x_2)^2-x_1-x_2)}\ d\mathbf{x} \approx \frac{(e-1)^2}{N}\sum_{i=1}^N e^{(X_{1i}+X_{2i})^2-X_{1i}-X_{2i}}.$$

```
N = 10000; U = rand(2,N); T = exp(sum(U).^2);
disp( [mean(T) 2*std(T)/sqrt(N)]) % simple MC
      4.9204      0.12261
e = exp(1); X = log(1+(e-1)*U); T = (e-1)^2*exp(sum(X).^2-sum(X));
disp( [mean(T) 2*std(T)/sqrt(N)]) % importance sampled MC
      4.8863     0.065169
```

Better $g(\mathbf{x}) = e^{2x_1}e^{2x_2}$, with $g(1,1) = f(1,1)$? Then $G_i(x) = \frac{e^{2x}-1}{e^2-1}$,
$X_{ij} = \ln(1+(e^2-1)U_{ij})/2$, and $\theta = \frac{(e^2-1)^2}{4}\int_0^1\int_0^1 \frac{4e^{2(x_1+x_2)}}{(e^2-1)^2}e^{((x_1+x_2)^2-2(x_1+x_2))}\ d\mathbf{x}$,

```
e = exp(1); X = log(1+(e^2-1)*U)/2;
T = (e^2-1)^2*exp(sum(X).^2-2*sum(X))/4;
disp( [mean(T) 2*std(T)/sqrt(N)])
      4.9008    0.0082436
```

Maybe even better $g(\mathbf{x}) = 1 + (x_1+x_2)^2$?

IMPORTANCE SAMPLING CONT.

Tilted Densities $g(x)$: given pdf $f(x)$, let $M(t) = \int e^{tx} f(x)dx$ (moment gen. fun.).
The **tilted density** for $f(x)$ is $f_t(x) = e^{tx} f(x)/M(t)$.

- Tilted Density Examples
  - Exponential densities: if $f(x) = \lambda e^{-\lambda x}$, $f_t(x) = (\lambda - t)e^{-(\lambda - t)x}$, for $t < \lambda$;

  - Bernoullli pmf's: $f(x) = p^x (1-p)^{1-x}$, $x = 0, 1$. $M(t) = E_f[e^{tx}] = e^t p + (1-p)$,
    so $f_t(x) = \frac{e^{tx} p^x (1-p)^{1-x}}{e^t p + (1-p)} = \left(\frac{e^t p}{e^t p + (1-p)}\right)^x \left(\frac{1-p}{e^t p + (1-p)}\right)^{1-x}$, a Bernoulli RV pmf,
    with $p_t = \frac{e^t p}{e^t p + (1-p)}$, and $\frac{f}{f_t} = \frac{e^t p + (1-p)}{e^{tx}} = e^{-tx}(e^t p + (1-p))$.

Generalization: if $f(x)$ is a binomial $Bin(n, p)$ pmf, then
$f_t(x)$ is $Bin(n, e^t p + 1 - p)$, with $M(t) = (e^t p + 1 - p)^n$.

IMPORTANCE SAMPLING CONT.

– Tilted Normal densities: if $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$,

then $$e^{tx}f(x) = \frac{e^{xt}e^{-x^2/2}}{\sqrt{2\pi}} = \frac{e^{-(x-t)^2/2}e^{-t^2/2}}{\sqrt{2\pi}}.$$

so $f_t(x) = \frac{e^{-(x-t)^2/2}}{\sqrt{2\pi}}$ is $Normal(t, 1)$, with $M(t) = e^{-t^2/2}$.

Generalization: if $f(x)$ is a $Normal(\mu, \sigma^2)$ pdf, then
$f_t(x)$ is a $Normal(\mu + \sigma^2 t, \sigma^2)$ pdf.

- Choosing t: pick t with small $Var(\frac{h(\mathbf{x})f(\mathbf{x})}{f_t(\mathbf{x})})$.
  Text heuristic for exponentials and Bernoullis:
  if $h = I\{\sum X_i > a\}$, choose $t = t^*$ with $E_{t^*}[\sum X_i] \approx a$.
  For normal densities, pick to match mode or mean of $f(x)h(x)$.

IMPORTANCE SAMPLING CONT.

- Tilted Density Examples:

1. Bernoulli RV Examples: if $X_i's$ are independent Bernoulli(p_i) RV's and $\theta = I\{\sum_{i=1}^n X_i > a\} = I\{S > a\}$.

$$\hat{\theta} = I\{S > a\}e^{-tS}\prod_{i=1}^{n}(e^t p_i + (1-p_i)), \text{ with } E_t[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n}\frac{e^t p_i}{e^t p_i + (1-p_i)}.$$

Example with $n = 20$, $p_i = .4$, $a = 16$; choose t so that $E_t[S] = 20\frac{.4e^t}{.4e^t+.6} = 16$, with solution $e^{t^*} = 6$; then $p_t = .8$, $e^{t^*}p + (1-p) = 3$, and estimator is $\hat{\theta} = I\{\sum X_i > a\}6^{-S}3^{20} = 3^{20-S}I\{\sum X_i > a\}/2^S$. Matlab test:

```
N = 100000; p = .4; n = 20; I = sum( rand(n,N) < p ) > 16;
disp([mean(I) 2*std(I)/sqrt(N)])   % Simple MC
   6e-05   4.8989e-05
S = sum( rand(n,N) < .8 ); I = 3.^(20-S).*( S > 16 )./2.^S;
disp([mean(I) 2*std(I)/sqrt(N)])   % Importance sampled MC
   4.7575e-05   5.1608e-07
N = 10000000; p = .4; n = 20; I = sum( rand(n,N) < p ) > 16;
disp([mean(I) 2*std(I)/sqrt(N)])   % Simple MC , 10M samples
   4.82e-05   4.3908e-06
```

Note: $\theta = \sum_{i=17}^{20}\binom{20}{i}(.4)^i(.6)^{20-i} \approx 4.7345 \times 10^{-5}$.

IMPORTANCE SAMPLING CONT.

2. Exponential RV Example: if $X_i \sim Exp(\frac{1}{i+2})$, $i = 1, 2, 3, 4$, $S(\mathbf{X}) = \sum_{i=1}^{4} X_i$,

$$\text{estimate} \quad \theta = \int_0^\infty \cdots \int_0^\infty \frac{e^{-\sum_{i=1}^4 \frac{x_i}{i+2}}}{3 \cdot 4 \cdot 5 \cdot 6} S(\mathbf{x}) I\{S(\mathbf{x}) > 62\}\, d\mathbf{x},$$

Raw MC uses $X_{ij} \sim Exp(\frac{1}{i+2})$, to estimate $\theta \approx \frac{1}{N}\sum_{j=1}^{N} h(\mathbf{X}_j)$,
with $h(\mathbf{X}) = S(\mathbf{X}) I\{S(\mathbf{X}) > 62\}$. Matlab:

```
N = 100000; U = rand(4,N);
X = -diag([3:6])*log(1-U);
S = sum(X); h = S.*( S > 62 );
disp( [mean(h) 2*std(h)/sqrt(N)])
     0.066974      0.013647
```

Note: to find $E[S|S > 62]$, divide by $E[I(S > 62)]$

```
E = h/mean((S>62));
disp([mean(E) 2*std(E)/sqrt(N)])
       66.948       15.237
```

IMPORTANCE SAMPLING CONT.

For tilted density, use common tilt parameter t, so that $X_i \sim Exp(1/(i+2)-t)$,

$$\theta = \prod_{i=1}^{4} \frac{i+2}{1-(i+2)t} \int_{[0,\infty)^4} \frac{e^{-\sum_{i=1}^4 x_i(\frac{1}{i+2}-t)}}{\prod_{i=1}^4 \frac{i+2}{1-(i+2)t)}} \frac{e^{-tS(\mathbf{x})}h(\mathbf{x})}{\prod_{i=1}^4 (i+2)} \, d\mathbf{x};$$

$$\approx \frac{C}{N} \sum_{j=1}^{N} h(\mathbf{X}_j) e^{-tS(\mathbf{X}_j)}, \text{ with } C = \prod_{i=1}^{4} \frac{1}{1-(i+2)t}.$$

Text estimates “good” $t = .14$, by approximately solving

$$\sum_{i=1}^{4} E_t[X_i] = \frac{3}{1-3t} + \frac{4}{1-4t} + \frac{5}{1-5t} + \frac{6}{1-6t} = 62.$$

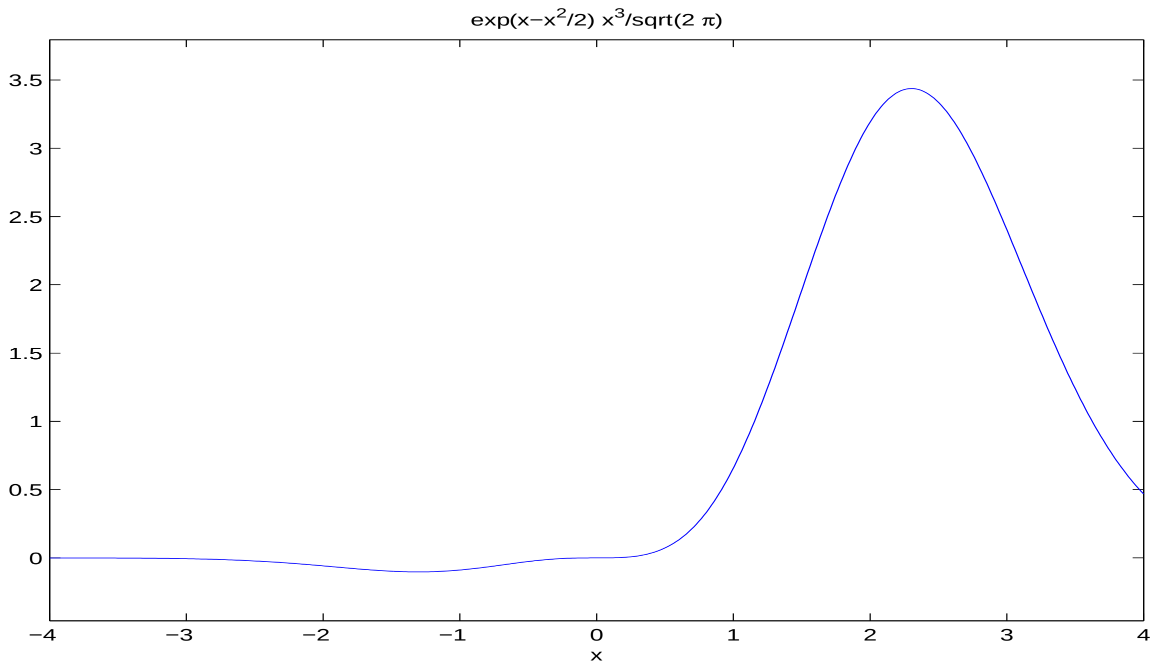
But “guess and check” with Matlab finds “better” $t \approx .136$. Matlab tests:

```
t = .14; Cd = 1./(1-[3:6]*t); C = prod(Cd);
St = -([3:6].*Cd)*log(1-U); ht = C*St.*( St>62 ).*exp(-t*St);
disp([mean(ht) 2*std(ht)/sqrt(N)])   0.063281    0.0010059
t = .136; Cd = 1./(1-[3:6]*t); C = prod(Cd);
St = -([3:6].*Cd)*log(1-U); ht = C*St.*( St>62 ).*exp(-t*St);
disp([mean(ht) 2*std(ht)/sqrt(N)])   0.06201   0.00099896
E = ht/mean(C*(St > 62).*exp(-t*St));
disp([mean(E) 2*std(E)/sqrt(N)])% Expected Value
      68.215       1.0931
```

IMPORTANCE SAMPLING CONT.

- Tilting for Normal Densities: if $f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}$,
  tilted density $f_t(x) = \frac{f(x)e^{xt}}{M(t)}$ is a shifted normal.

  Example: $\theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} z^3 e^z dz$. Pick t to match point (mode) where integrand $a(z) = Ke^{-(z-1)^2/2} z^3$ is max.



$a'(z) = (3/z - (z-1))a(z) = 0$, at $z = 0, (1 \pm \sqrt{13})/2$,
with global maximum at $t = (1 + \sqrt{13})/2 \approx 2.3$.

IMPORTANCE SAMPLING CONT.

Or, pick t to match mean at $t = 2.5$.

For either case (using $z = w + t$) :

$$\theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-t)^2/2} z^3 e^{z-zt+t^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2/2} (w+t)^3 e^{(w+t)(1-t)+t^2/2)} dt.$$

Matlab tests:

```
N = 100000; W = randn(1,N); t = (1+sqrt(13)/2; % ~ 2.3
Y = (W+t).^3.*exp((W+t)*(1-t)+t^2/2);
disp( [mean(Y) 2*std(Y)/sqrt(N)]) % mode tilt
       6.6408     0.035346
t = 2.5; Y = (W+t).^3.*exp((W+t)*(1-t)+t^2/2);
disp( [mean(Y) 2*std(Y)/sqrt(N)]) % mean tilt
       6.6571     0.037741
```

Compare:

```
t = 1; Y = (W+t).^3.*exp((W+t)*(1-t)+t^2/2);
disp( [mean(Y) 2*std(Y)/sqrt(N)]) % tilt = 1
       6.5775     0.079949
t=0; Y = (W+t).^3.*exp((W+t)*(1-t)+t^2/2);
disp( [mean(Y) 2*std(Y)/sqrt(N) ]) % t=0, raw MC
       6.3272      0.29958
```

IMPORTANCE SAMPLING CONT.

- Tilting for Multidimensional Normal Density Problems: use vector $\mathbf{t}$.
  Choice of $\mathbf{t}$? Try to make $Var(\frac{h(\mathbf{x})f(\mathbf{x})}{f(\mathbf{x}-\mathbf{t})})$ small:
  a) choose point $\mathbf{t}$ where $h(\mathbf{x})f(\mathbf{x})$ is maximum (mode), or
  b) choose $\mathbf{t} = E[\mathbf{x}h(\mathbf{x})]/E[h(\mathbf{x})]$ (mean).

  Asian Option example: this has $S_m = S_{m-1}e^{(r-\frac{\sigma^2}{2})\delta+\sigma\sqrt{\delta}Z_m}$,
  with $\delta = T/M$, $Z_m \sim Normal(0,1)$ and expected profit

$$\theta = E[e^{-rT}\max(\frac{1}{M}\sum_{i=1}^{M} S_i(\mathbf{Z}) - K, 0)] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{e^{-\sum_{i=1}^{M} z_i^2/2}}{(\sqrt{2\pi})^M}h(\mathbf{z})d\mathbf{z},$$

  with $h(\mathbf{z}) = e^{-rT}\max(\frac{1}{M}\sum_{i=1}^{M} S_i(\mathbf{Z}) - K, 0)$.
  For (mode) method a), find $\mathbf{t}$ to maximize $h(\mathbf{z})e^{-\sum_{i=1}^{M} z_i^2/2}$.
  For (mean) method b), $\mathbf{t}$ can be estimated from raw MC data.
  Given some $\mathbf{t}$, use

$$\begin{aligned}\hat{\theta} &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{e^{-\sum_{i=1}^{M}(z_i-t_i)^2/2}}{(\sqrt{2\pi})^m}\frac{e^{-\sum_{i=1}^{M} z_i^2/2}}{e^{-\sum_{i=1}^{M}(z_i-t_i)^2/2}}h(\mathbf{z})\, d\mathbf{z}\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{e^{-\sum_{i=1}^{M} y_i^2/2}}{(\sqrt{2\pi})^M}\ \frac{e^{-\sum_{i=1}^{M}(y_i+t_i)^2/2}}{e^{-\sum_{i=1}^{M} y_i^2/2}}h(\mathbf{y}+\mathbf{t})\, d\mathbf{y}.\end{aligned}$$

IMPORTANCE SAMPLING CONT.

Example with $M = 16,\ S_0 = K = 50,\ T = 1,\ r = .05,\ \sigma = .1.$
Matlab test using method b):

```
M = 16; S0 = 50; K = 50; T = 1; dlt = T/M;
r = 0.05; s = 0.1; rd = ( r - s^2/2 )*dlt;
N = 10000; z = randn(M,N); % Simple MC
S = S0*exp(cumsum(rd + s*sqrt(dlt)*z));
h = exp(-r*T)*max( mean(S)-K, 0 );
disp([mean(h) var(h) 2*std(h)/sqrt(N)])  % Raw MC
      1.9465       4.825       0.043932
t = z*h'/sum(h); % compute approximate mean tilt vector
y = z; z = y + t*ones(1,N);
S = S0*exp(cumsum(rd + s*sqrt(dlt)*z));
h = exp(-r*T)*max( mean(S)-K, 0 );
ht = h.*exp(sum(y.*y-z.*z)/2); % Importance Sampling
disp([mean(ht) var(ht) 2*std(ht)/sqrt(N)])
      1.9136      0.66366     0.016293
disp(t')
0.45019 0.41587 0.36157 0.37366 0.32656 0.29464  0.29179  0.26766
0.23193 0.22963 0.15875 0.14859 0.1231  0.085523 0.084035 0.04595
```

Notice variance reduction from mean tilted sampling.

IMPORTANCE SAMPLING CONT.

Asian Option example with $M = 16,\ S_0 = K = 50,\ T = 1,\ r = .05,\ \sigma = .1.$
Using method a) with Matlab "fminsearch" to find t:

```
Sf = @(z)S0*exp(cumsum(rd+s*sqrt(dlt)*z));
hf = @(z)exp(-z'*z/2)*max(mean(Sf(z))-K,0);
t = fminsearch(@(z)-hf(z), t ); % Tilt t
y = z; z = y + t*ones(1,N);
S = S0*exp(cumsum(rd + s*sqrt(dlt)*z));
h = exp(-r*T)*max( mean(S)-K, 0 );
ht = h.*exp(sum(y.*y-z.*z)/2); % Importance
disp([mean(ht) var(ht) 2*std(ht)/sqrt(N)])
      1.8868      0.35124      0.011853
disp(t')
  0.34554 0.31453 0.30022 0.2869  0.27423  0.24198 0.21827 0.20444
  0.1726  0.15727 0.11179 0.11349 0.088588 0.06109 0.06342 0.05281
```

Notice additional variance reduction from mode tilted sampling.

IMPORTANCE SAMPLING CONT.

Asian Option example with $M = 200,\ S_0 = K = 50,\ T = 1,\ r = .05,\ \sigma = .1.$

```
M = 200; S0 = 50; K = 50; T = 1; dlt = T/M;
r = 0.05; s = 0.1; rd = ( r - s^2/2 )*dlt;
N = 10000; z = randn(M,N); % Simple MC
S = S0*exp(cumsum(rd + s*sqrt(dlt)*z));
h = exp(-r*T)*max( mean(S)-K, 0 );
disp([mean(h) var(h) 2*std(h)/sqrt(N)])  % Raw MC
     1.8423       4.4635     0.042254
t = z*h'/sum(h); % compute approximate Mean Tilt Vector
y = z; z = y + t*ones(1,N);
S = S0*exp(cumsum(rd + s*sqrt(dlt)*z));
h = exp(-r*T)*max( mean(S)-K, 0 );
ht = h.*exp(sum(y.*y-z.*z)/2); % Importance Sampling
disp([mean(ht) var(ht) 2*std(ht)/sqrt(N)])
 1.7976      0.77736     0.017634
```