VARIANCE REDUCTION from CONDITIONING

Conditioning Background: suppose $\theta = E[X]$, but X simulation depends on some other RV Y. Recall $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$, so $Var(E[X|Y]) \le Var(X)$; but $\theta = E[E[X|Y]]$, and so simulation of $E[X|Y]$ should reduce variance.

Conditioning Variance Reduction Examples

1. Estimation of $\pi/4$: $\frac{\pi}{4} = P\{V_1^2 + V_2^2 < 1\} = P\{I\}$,
   $V_1, V_2 \sim Uniform(-1,1)$, with $I = \begin{cases} 1 & \text{if } V_1^2 + V_2^2 < 1 \\ 0 & \text{otherwise} \end{cases}$
   Conditioning on V_1
   $$P\{I|V_1 = v\} = P\{v^2 + V_2^2 < 1|V_1 = v\} = P\{V_2^2 < 1 - v^2\}$$
   $$= \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{1}{2}\ dx = \sqrt{1-v^2}.$$
   So $E[I|V_1] = \sqrt{1-V_1^2}$, with $Var(\sqrt{1-V_1^2}) \approx .0498$, $Var(I) = \frac{\pi}{4}(1-\frac{\pi}{4}) \approx .1686$.
   Some Matlab results

```
N = 1000; U = rand(2,N); V = 2*U - 1; I = sum( V.^2 ) < 1;
disp( [mean(I) var(I) 2*std(I)/sqrt(N)] )   % simple MC
   0.786       0.16837      0.025952
IV = sqrt(1-V(1,:).^2);
disp( [mean(IV) var(IV) 2*std(IV)/sqrt(N)]) % conditioned MC
   0.78089     0.048267      0.013895
```

CONDITIONED VAR REDUCTION CONT.

2. Estimation of $p = P\{\sum_{i=1}^{3} iX_i \geq 2\}$, with $X_i \sim Exp(1)$. Writing p as an integral

$$p = \int_0^\infty \int_0^\infty \int_0^\infty I(\sum_{i=1}^{3} ix_i \geq 2)e^{-x_1}e^{-x_2}e^{-x_3}dx_3dx_2dx_1,$$

but $1-p$ is $P\{\sum_{i=1}^{3} iX_i < 2\}$, with $X_i \sim Exp(1)$, so

$$1-p = \int_0^\infty \int_0^\infty \int_0^\infty I(\sum_{i=1}^{3} ix_i < 2)e^{-x_1}e^{-x_2}e^{-x_3}dx_3dx_2dx_1,$$

an integral over the tetrahedron bounded by x_i axes, and $x_1 + 2x_2 + 3x_3 = 2$.
Note: simple MC would use $X_i = -\ln(U_i)$ and RV $I = (\sum_{i=1}^{3} iX_i < 2)$.
Conditioning on X_1,

$$E[I] = E[E[I|X_1 = x_1]] = E[E[2x_2 + 3x_3 < 2 - x_1|x_1 < 2]].$$

So $1-p = \int\limits_0^2 e^{-x_1} \int\limits_0^\infty \int\limits_0^\infty I(2x_2 + 3x_3 < 2 - x_1)e^{-x_2}e^{-x_3}\ dx_3dx_2dx_1$

CONDITIONING VAR REDUCTION CONT.

You need to generate random X_1's so you need the conditional cdf for X_1.

If $(1-e^{-x_1})/(1-e^{-2}) = u$, with $e^{-x_1}dx_1 = (1-e^{-2})du$;
then use $X_1 = -\ln(1-(1-e^{-2})U)$, so

$$1-p = (1-e^{-2})\int_0^1\int_0^\infty\int_0^\infty I(2x_2+3x_3 < 2-x_1(u))e^{-x_2}e^{-x_3}\,dx_3dx_2du$$

For conditional simulation use
$X_1 = -\ln(1-U_1(1-e^{-2}))$, $X_2 = -\ln(1-U_2)$ and $X_3 = -\ln(1-U_3)$
Some Matlab results

```
N = 100000; U = rand(3,N); X = -log(1-U); I = [1 2 3]*X < 2;
disp( [1-mean(I) var(I) 2*std(I)/sqrt(N)] ) % simple MC
    0.90705      0.084311     0.0018364
X(1,:) = -log( 1-U(1,:)*(1-exp(-2)) );
I = (1-exp(-2))*([1 2 3]*X < 2);
disp( [1-mean(I) var(I) 2*std(I)/sqrt(N)] ) % conditional MC
    0.90703      0.071744      0.001694
```

CONDITIONING VAR REDUCTION CONT.

Conditioning could also be used for x_2 and x_3:

$$\begin{aligned} E[I] &= E[E[I|X_1 = x_1]] = E[E[2x_2 + 3x_3 < 2 - x_1 | x_1 < 2]] \\ &= E[E[E[3x_3 < 2 - x_1 - 2x_2 | x_2 < (2 - x_1)/2] | x_1 < 2]]; \end{aligned}$$

$$1 - p = \int_0^2 e^{-x_1} \int_0^{(2-x_1)/2} e^{-x_2} \int_0^{(2-x_1-2x_2)/3} e^{-x_3}\, dx_3 dx_2 dx_1,$$

Conditional cdfs are $\frac{1-e^{-x_1}}{1-e^{-2}} = u_1$, with $e^{-x_1}dx_1 = (1-e^{-2})du_1$,
$\frac{1-e^{-x_2}}{1-e^{-(2-x_1)/2}} = u_2$, with $e^{-x_2}dx_2 = (1-e^{-(2-x_1)/2})du_2$,
$\frac{1-e^{-x_3}}{1-e^{-(2-x_1-2x_2)/3}} = u_3$, with $e^{-x_3}dx_3 = (1-e^{-(2-x_1-2x_2)/3})du_3$.

$$1 - p = (1-e^{-2}) \int_0^1 (1 - e^{-(2-x_1(u_1))/2}) \int_0^1 (1 - e^{-(2-x_1(u_1)-2x_2(u_2))/3}) \int_0^1 du_3 du_2 du_1.$$

CONDITIONED VAR REDUCTION CONT.

$$1-p=(1-e^{-2})\int_0^1(1-e^{-(2-x_1(u_1))/2})\int_0^1(1-e^{-(2-x_1(u_1)-2x_2(u_2))/3})\int_0^1 du_3du_2du_1.$$

The last variable is not needed; for conditional simulation use
$X_1=-\ln(1-U_1(1-e^{-2}))$, $X_2=-\ln(1-U_2(1-e^{-(2-X_1)/2}))$,
with RV $Z=(1-e^{-2})(1-e^{-(2-X_1)/2})(1-e^{-(2-X_1-2X_2)/3})$.

Some Matlab results

```
N = 100000; U = rand(3,N); X = -log(1-U); I = [1 2 3]*X < 2;
disp( [1-mean(I) var(I) 2*std(I)/sqrt(N)] )   % simple MC
      0.90717     0.084213    0.0018354
X(1,:) = -log( 1-U(1,:).*(1-exp(-2)) );
X(2,:) = -log( 1-U(2,:).*(1-exp(X(1,:)/2-1)) );
Z = ( 1 - exp(-2) )*( 1 - exp(X(1,:)/2-1) );
Z = Z.*( 1 - exp( (X(1,:)+2*X(2,:)-2)/3 ) );
disp( [1-mean(Z) var(Z) 2*std(Z)/sqrt(N)] )
      0.90643    0.0049437   0.00044469
```

CONDITIONED VAR REDUCTION CONT.

3. Suppose $Y \sim Exp(1)$. Given $Y = y$, $X \sim Normal(y, 4)$: find $p = P\{X > 1\}$ (c.p. text problem 9.18). "Raw simulation" would count proportion of times $X = 2Z - \ln(U) > 1$, with $Z \sim Normal(0, 1)$.

For conditioning, consider p as an integral $p = \int_0^\infty e^{-y}\Big(\int_1^\infty \frac{e^{-\frac{(x-y)^2}{2(4)}}}{2\sqrt{2\pi}}dx\Big)dy$.
If (standardized) variable $z = (x - y)/2$ (with $dz = dx/2$) is used

$$p = \int_0^\infty e^{-y}\Big(\int_{\frac{1-y}{2}}^\infty \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\Big)dy = \int_0^\infty e^{-y}\Big(1 - \Phi\Big(\frac{1-y}{2}\Big)\Big)dy.$$

Conditional simulation computes
$E[I|Y] = P\{X > 1|Y = y\} = P\{Z > \frac{1-y}{2}\} = 1 - \Phi(\frac{1-y}{2})$.
Matlab results including antithetic variates

```
N = 1000; U = rand(1,N); Y = -log(U); I = Y+2*randn(1,N) > 1;
disp( [mean(I) var(I) 2*std(I)/sqrt(N)] )  % simple MC
    0.488      0.25011     0.031629
W = 1 - normcdf((1-Y)/2);
disp( [mean(W) var(W) 2*std(W)/sqrt(N)] ) % conditioned MC
    0.48532      0.02679      0.010352
A = ( W + 1 - normcdf((1+log(1-U))/2) )/2;
disp( [mean(A) var(A) 2*std(A)/sqrt(N)] ) % with antithetic vars
    0.49081     0.003109     0.0035265
```

CONDITIONED VAR REDUCTION CONT.

4. Asian option: this has $S_m = S_{m-1}e^{(r-\frac{\sigma^2}{2})\delta+\sigma\sqrt{\delta}Z}$, with $\delta = T/M$, $Z \sim Normal(0,1)$ and expected profit $P = E[e^{-rT}\max(\frac{1}{M}\sum_{i=1}^{M} S_i(\mathbf{Z}) - K, 0)]$. Written as an integral

$$P = \frac{e^{-rT}}{(2\pi)^{M/2}} \int_{-\infty}^{\infty} e^{-\frac{z_1^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z_2^2}{2}} \cdots \int_{-\infty}^{\infty} e^{-\frac{z_M^2}{2}} \ \ \max(\frac{1}{M}\sum_{i=1}^{M} S_i(\mathbf{z}) - K, 0) \ \ d\mathbf{z}.$$

Conditioning can use the integration region constraint $\sum_{i=1}^{M} S_i(\mathbf{Z}) > MK$. Let $a = (r - \frac{\sigma^2}{2})\delta, b = \sigma\sqrt{\delta}$, and $T_m(\mathbf{z}) = S_0 \sum_{i=1}^{m} \prod_{j=1}^{i} e^{a+bz_i}$, so constraint on last variable is

$$T_{M-1}(\mathbf{z}) + S_{M-1}(\mathbf{z})e^{a+bz_M} > MK;$$

$$e^{a+bz_M} > (MK - T_{M-1}(\mathbf{z}))/S_{M-1}(\mathbf{z});$$

$$z_M > z^* = \Big(\ln(max((MK - T_{M-1}(\mathbf{z}))/S_{M-1}(\mathbf{z}), 0) - a \Big)/b;$$

and therefore

$$P = \frac{e^{-rT}}{(2\pi)^{M/2}} \int_{-\infty}^{\infty} e^{-\frac{z_1^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z_2^2}{2}} \cdots \int_{z^*}^{\infty} e^{-\frac{z_M^2}{2}} \ \ (\frac{T_M(\mathbf{z})}{M} - K) \ d\mathbf{z}.$$

Innermost integral can be computed by formula; conditioning reduces P to an $(M-1)$-dimensional integral.

CONDITIONED VAR REDUCTION CONT.

5. Simulating a single server Q system with n arrivals, and server time $\sim Exp(\frac{1}{\mu})$.
If customer i time-in-system is W_i, want to estimate $\theta = E[\sum_{i=1}^n W_i]$.
If S_i is the "system state" when customer i arrives,

$$E\left[\sum_{i=1}^{n} E[W_i|S_i]\right] = \sum_{i=1}^{n} E[E[W_i|S_i]] = E[\sum_{i=1}^{n} W_i] = \theta.$$

If $S_i = N_i$ # customers in system, $E[W_i|S_i] = E[W_i|N_i] = N_i\mu$.
So use $\theta = E[\sum_{i=1}^n N_i\mu] = \mu E[\sum_{i=1}^n N_i]$.
Can modify single-server program to output W's and N's.
Tests with total customers = 10, arrival rate = 2, $\mu = 1$ or $1/2$.

```
K = 10000;
for i = 1:K, [W N] = snglsvc(10,2,1);
  X(i) = sum(W); Y(i) = sum(N);
end, disp([mean(X) var(X) mean(Y) var(Y)])
35.73    363.91    35.725    102.26
for i = 1:K, [W N] = snglsvc(10,2,2);
  X(i) = sum(W); Y(i) = sum(N)/2;
end, disp([mean(X) var(X) mean(Y) var(Y)])
12.363    62.183    12.374    21.596
```

Notice smaller var(Y)'s (from $\mu\sum_{i=1}^n N_i$'s) than var(X)'s.

CONDITIONED VAR REDUCTION CONT.

```
function [W,N] = snglsvc( C, la, ls )
% Single-Server Q Simulation, for
%   C customers, Exp(la) interarrivals
%  Output is W (wait times), N (#'s of customers)
  t = 0; na = 0; nd = 0; n = 0;
  ta = E(la); td = inf;
  while na < C % more arrivals permitted
    if ta <= td, t = ta; n = n + 1; % new arrival
      na = na + 1; A(na) = t;
      ta = t + E(la); N(na) = n; % collect N
      if n == 1, td = t + E(ls); end,
    else % departure
      t = td; n = n - 1; nd = nd + 1; D(nd) = t;
      if n > 0, td = t + E(ls); else td = inf; end
    end
  end % no more arrivals, empty the Q
  while n > 0, t = td; nd = nd + 1; D(nd) = t;
    n = n - 1; td = t + E(ls);
  end, W = D - A;
% end snglsv
function Y = E(lam), Y = -log(rand)/lam;
```

CONDITIONED VAR REDUCTION CONT.

6. Simulating a sum of N iid RVs X_i when N is RV: estimate $p = P\{\sum_{i=1}^{N} X_i > c\}$.
   E.g. X_i is insurance claim i, N is # claims by time T.
   Note: $S = \sum_{i=1}^{N} X_i$, called a **compound** RV.
   In many cases, distribution for N is known, e.g. Poisson.
   "Raw simulation": for K runs, generate N and N X_is;
   then count proportion with $\sum_{i=1}^{N} X_i > c$.
   Conditional simulation:
   let $M = min(n : \sum_{i=1}^{n} X_i > c)$; and use $p = E[P\{N \geq M | M = m\}]$,
   because $E[\sum_{i=1}^{N} X_i > c | M] = P\{N \geq M | M\} = p$.
   Conditional simulation generates X_is until $\sum_{i=1}^{m} X_i > c$ and then
   computes the p estimator using $P\{M \geq m\}$ from the N distribution.
   Note: written as an integral-sum when $N \sim Poisson(\delta)$, $X_i \sim Exp(\frac{1}{\lambda})$.

$$p = \sum_{n=1}^{\infty} \frac{e^{-\delta}\delta^n}{n!} \lambda^n \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} e^{-\lambda \sum_{i=1}^{n} x_i} I(\sum_{i=1}^{n} x_i > c) \;\; d\mathbf{x}_n,$$

$$1 - p = \sum_{n=1}^{\infty} \frac{e^{-\delta}\delta^n}{n!} \lambda^n \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} e^{-\lambda \sum_{i=1}^{n} x_i} I(\sum_{i=1}^{n} x_i \leq c) \, d\mathbf{x}_n,$$

with $d\mathbf{x}_n = dx_n \cdots dx_2 dx_1$.

CONDITIONED VAR REDUCTION CONT.

Example:

Insurance claims, mean 10/day, have $N \sim Poisson(10)$, with pmf $p_n = e^{-10}10^n/n!$.
If claims are $Exp(1/v)$, $v = 1 \times \$1000$, compute $P\{S > 15\}$ (with S in \$1000's).

```
K = 1000;
for i = 1 : K, N = poissrnd(10);
  I(i) = sum( -log(rand(1,N)) ) > 15;
end
disp( [ mean(I) std(I) 2*std(I)/sqrt(K) ] )  % Simple MC
     0.13422      0.34089      0.02156
for i = 1 : K, S = 0; N = 0;
  while S < 15, S = S - log(rand); N = N+1; end, M(i) = N-1;
end
W = 1 - poisscdf(M,10);
disp( [ mean(W) std(W) 2*std(W)/sqrt(K) ] )  % Conditioned MC
      0.13749      0.19047      0.012046
```

Further variance reduction using control variates:

if $\mu = E[X_i]$, use the control variate $Y = \sum_{i=1}^{M}(X_i - \mu)$.