ANTITHETIC VARIABLES, CONTROL VARIATES

Variance Reduction Background: the simulation error estimates for some parameter $\theta \approx \bar{X}$, depend on $Var(\bar{X}) = Var(X)/n$, so the simulation can be more efficient if $Var(X)$ is reduced.

Antithetic Variables

• Key idea: if $X_1$ and $X_2$ are id RVs with mean $\theta$, 
$$Var(\frac{X_1 + X_2}{2}) = \frac{1}{4}(Var(X_1) + Var(X_2) + 2 Cov(X_1, X_2)),$$
so variance is reduced if $X_1$ and $X_2$ have $Cov(X_1, X_2) \leq 0$.

• For many simulations, a $\theta$ estimator is $X_1 = h(U_1, \ldots, U_n)$ (with uniform $U_i$’s) for some $h$; so consider the antithetic estimator $X_2 = h(1-U_1, \ldots, 1-U_n)$. The combined estimator is $(X_1 + X_2)/2$.

a) the simplest example has $h(x) = x$, so if $X_1 = U$, then $X_2 = 1-U$, and 
$$(X_1 + X_2)/2 = 1/2,$$ 
with $Var((X_1 + X_2)/2) = 0$ (perfect negative correlation).

b) Theorem: if $h(X_1, \ldots, X_n)$ is monotone for each variable, then 
$$Cov(h(U_1, \ldots, U_n), h(1-U_1, \ldots, 1-U_n)) \leq 0.$$ 

c) if $h(Y)$ is monotone, and $Y_i$’s are not iid uniform, but $Y_i = F_i^{-1}(U_i)$, $i = 1, \ldots, n$, then 
$$X(U) = h(F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n))$$ 
is monotone in $U_i$’s, so use 
$$W = (X(U) + X(1-U))/2.$$
ANTITHETIC VARIABLES CONTINUED

Antithetic Variable Examples

1. Estimate the integral $V = \int_{0}^{\infty} e^{-x} \ln(1 + x^2) \, dx$ with MC. Use $t = 1 - e^{-x}$:

Matlab for MC method with antithetic variates:

```matlab
N = 1000; g = @(t)log(1+log(1-t).^2); T = rand(1,N); X = g(T);
disp([mean(X) std(X) 2*std(X)/sqrt(N)]) % simple MC
    0.66848   0.73278   0.046345
T = rand(1,N/2); X = ( g(T) + g(1-T) )/2;
disp([mean(X) std(X) 2*std(X)/sqrt(N/2)]) % antithetic vars
    0.67881   0.30933   0.027667
```

Could also use $x = t/(1 - t)$, with $dx = dt/(1 - t)^2$.

Matlab for MC method with antithetic variates:

```matlab
f = @(x)log(1+x.^2).*exp(-x); h = @(t)f(t./(1-t))./(1-t).^2;
T = rand(1,N); X = h(T);
disp([mean(X) std(X) 2*std(X)/sqrt(N)]) % simple MC
    0.6695   0.71206   0.045034
T = rand(1,N/2); X = ( h(T) + h(1-T) )/2;
disp([mean(X) std(X) 2*std(X)/sqrt(N/2)]) % antithetic vars
    0.67492   0.46416   0.041516
```
2. Estimate \( V = \int_{0}^{\pi/4} \int_{0}^{\pi/4} x^2y^2 \sin(x + y) \ln(x + y) \, dx \, dy \) with MC; notice integrand is monotone increasing in \( x \) and \( y \). Use \( x = \pi u_1/4 \), \( y = \pi u_2/4 \):

Matlab for MC method with antithetic variates:

\[
\begin{align*}
f &= @(x) \text{prod}(x)^2 \cdot \sin(\text{sum}(x)) \cdot \log(\text{sum}(x)); \\
N &= 1000; U = \text{rand}(2,N); X = \pi^2 f(\pi*U/4)/16; \\
\text{disp}([\text{mean}(X) \, \text{std}(X) \, 2*\text{std}(X)/\sqrt{N}]) \quad \% \text{simple MC} \\
&\quad 0.0037452 \quad 0.012534 \quad 0.00079271 \\
U &= \text{rand}(2,N/2); X = \pi^2 ((f(\pi*U/4) + f(\pi*(1-U)/4))/2)/16; \\
\text{disp}([\text{mean}(X) \, \text{std}(X) \, 2*\text{std}(X)/\sqrt{N/2}]) \quad \% \text{antithetic vars} \\
&\quad 0.0037885 \quad 0.0085332 \quad 0.00076323
\end{align*}
\]
ANTITHETIC VARIABLE EXAMPLES CONT.

3. Simulating a queueing system with \( N \) arrivals.

Two-servers in parallel example.

Inter-arrival times for \( N \) arrivals are \(-\ln(1 - U_i)/\lambda\), and server times
are \( G_1^{-1}(V_i) \) or \( G_2^{-1}(V_i) \) for \( i = 1, \ldots, N \), with \( U_i, V_i \sim \text{Uniform}(0, 1) \).

Output (e.g. average time in system) is a function \( D(U_1, \ldots, U_N, V_1, \ldots, V_N) \),
which should be monotone increasing in \( U \)s and \( V \)s.

Matlab results for 2-server parallel queue, ave. time in system,
with “prllsv” function modified to allow \( U \)’s and \( V \)’s as inputs:

\[
N = 1000; M = 100;
\]

for \( i = 1 : M \), \( U = \text{rand}(1,N) \); \( V = \text{rand}(1,N) \);

\[ D(i) = \text{prllsvp}(N,U,V,6,4,3); \]
end

disp([mean(D) std(D) 2*std(D)/sqrt(M)]) \% simple MC

\[
0.99208 \quad 0.36936 \quad 0.073871
\]

for \( i = 1 : M/2 \), \( U = \text{rand}(1,N) \); \( V = \text{rand}(1,N) \);

\[ A(i)= \left( \text{prllsvp}(N,U,V,6,4,3) + \text{prllsvp}(N,1-U,1-V,6,4,3) \right)/2; \]
end

disp([mean(A) std(A) 2*std(A)/sqrt(M/2)])

\[
1.0254 \quad 0.20094 \quad 0.056835
\]
CONTROL VARIATES

Control Variates

- Background: assume the desired simulation quantity is $\theta = E[X]$;
  but there is another simulation RV $Y$ with known $\mu_Y = E[Y]$.

For any $c$, the RV $Z = X + c(Y - \mu_Y)$, is an unbiased estimator of $\theta$,
because $E[Z] = E[X] + c(E[Y] - \mu_Y) = \theta$.

Now $Var(Z) = Var(X + cY) = Var(X) + c^2 Var(Y) + (2c) Cov(X, Y)$
is minimized when $c = c^* = -Cov(X, Y)/Var(Y)$, and

$$Var(Z) = Var(X + c^*Y) = Var(X) - Cov(X, Y)^2/Var(Y).$$

$Y$ is called a control variate for $X$, with $Var(Z) \leq Var(X)$.
In order to enhance variance reduction, choose a $Y$ correlated with $X$.

- Implementation: $Cov(X, Y)$, $Var(Y)$ and $c^*$, are estimated from the data.

$$Cov(X, Y) \approx \hat{Cov}(X, Y) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}),$$

$$Var(Y) \approx \hat{Var}(Y) = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2, \quad c^* \approx \hat{c}^* = -\frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2},$$

- Note: the goal is to choose $Y$ so that $Y$ is correlated with $X$,
  with $Y$ easy to simulate and $\mu_Y$ easy to find.
Control Variate Examples

1. Simple example: to estimate \( \theta = \int_0^1 e^x \, dx \),
   
   try \( Y = U \), so \( \mu_Y = 1/2, Var(U) = 1/12 \); also
   
   \[
   Var(e^U) = \int_0^1 e^{2x} \, dx - (\int_0^1 e^x \, dx)^2 = (e^2 - 1)/2 - (e - 1)^2 \approx 0.242;
   \]
   
   \[
   Cov(e^U, U) = \int_0^1 x e^x \, dx - \int_0^1 x \, dx \int_0^1 e^x \, dx = 1 - (e - 1)/2 \approx 0.141086
   \]
   
   \[
   Var(X + c*Y) = Var(X) - Cov(X, Y)^2/Var(Y) \approx 0.0039.
   \]

Matlab for MC method with control variates:

\[
\begin{align*}
N &= 1000; U = \text{rand}(1,N); X = \exp(U); \\
\text{disp}([\text{mean}(X) \ \text{std}(X) \ 2*\text{std}(X)/\sqrt{N}]) &\quad \% \text{simple MC} \\
&\quad 1.7488 \quad 0.49766 \quad 0.031475 \\
Y &= U; \mu_Y = 1/2; Xb = \text{mean}(X); Yb = \text{mean}(Y); \\
cs &= -\text{sum}( (X-Xb).*(Y-Yb) )/\text{sum}( (Y-Yb).^2 ); \\
Z &= X + cs*( Y - \mu_Y ); \\
\text{disp}([\text{mean}(Z) \ \text{std}(Z) \ 2*\text{std}(Z)/\sqrt{N}]) &\quad \% \text{control variate MC} \\
&\quad 1.7189 \quad 0.063119 \quad 0.003992
\end{align*}
\]
CONTROL VARIATES CONTINUED

2. Estimate $V = \int_0^2 e^{-x^2} \, dx = 2 \int_0^1 e^{-(2u)^2} \, du$. Try $Y = 2e^{-2U}$:

Matlab for MC method with control variates:

```matlab
N = 1000; U = rand(1,N); X = 2*exp(-(2*U).^2);
disp([mean(X) std(X) 2*std(X)/sqrt(N)]) % simple MC
    0.91666    0.68842    0.04354
Y = 2*exp(-2*U); muY = 1 - exp(-2);
Xb = mean(X); Yb = mean(Y);
cs = -sum( (X-Xb).*(Y-Yb) )/sum( (Y-Yb).^2 );
Z = X + cs*( Y - muY );
disp([mean(Z) std(Z) 2*std(Z)/sqrt(N)]) % control variate MC
    0.88474    0.13402    0.0084762
```

Note: actual $V \approx 0.88208$. 

CONTROL VARIATE EXAMPLES CONT.

3. Estimate \( V = \int_0^{\pi/4} \int_0^{\pi/4} x^2 y^2 \sin(x + y) \ln(x + y) \, dx \, dy; \)
   Use \( x^2 y^2 \) to "approximate" the integrand.

Matlab for MC method with control variates:

```matlab
N = 1000; U = rand(2,N);
f = @(x)prod(x).^2.*sin(sum(x)).*log(sum(x));
X = pi^2*f(pi*U/4)/16; Xb = mean(X); % simple MC
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
    0.0035267   0.011023   0.00069718

g = @(x)prod(x).^2; muY = (pi/4)^6/9;
Y = pi^2*g(pi*U/4)/16; Yb = mean(Y);
cs = -sum((X-Xb).*(Y-Yb))/sum((Y-Yb).^2);
Z = X + cs*(Y - muY);
disp([mean(Z) std(Z) 2*std(Z)/sqrt(N)]) % control variate MC
    0.0037289   0.0043002   0.00027197
```
4. Asian option: this has $S_m = S_{m-1} e^{(r-s^2/6)\delta + \sigma \sqrt{\delta} Z}$, with $\delta = T/M$,

$Z \sim Normal(0, 1)$ and expected profit $E[e^{-rT} \max(\frac{1}{M} \sum_{i=1}^{M} S_i(Z) - K, 0)]$.

Control variate is $Y = e^{-rT} \max((\prod_{i=0}^{M} S_i(Z))^{1/(M+1)} - K, 0)$,

with $\mu_Y = e^{-rT} (e^{(r-s^2/6)\frac{T}{2}} S_0 \Phi(d) - K \Phi(d - \sigma \sqrt{\frac{T}{3}}))$,

$$d = (\ln(S_0/K) + (r + \sigma^2/6)T/2)/(\sigma \sqrt{T/3}).$$

Matlab for MC method with control variates:

```matlab
S0 = 50; K = 50; M = 16; T = 1; dlt = T/M; r = 0.05; s = 0.1;
rd = (r - s^2/2)*dlt; N = 10000; Z = randn(M,N);
S = S0*exp(cumsum(rd + s*sqrt(dlt)*Z)); % simple MC
X = exp(-r*T)*max( mean(S)-K, 0 );
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
   1.9522  2.2445  0.044889

Y = exp(-r*T)*max((S0*prod(S)).^(1/(M+1))-K,0);
Xb = mean(X); Yb = mean(Y); cs = -sum((X-Xb).* (Y-Yb))/sum((Y-Yb).^2)
muY = exp(-r*T)*( S0*exp((r-s^2/6)*T/2)*normcdf(d) ... 
               - K*normcdf(d-s*sqrt(T/3)) );

C = X + cs*( Y - muY );
disp([mean(C) std(C) 2*std(C)/sqrt(N)]) % control variate MC
   1.9375  0.03389  0.0006778
```
Matlab for MC method with control variates for M = 200:

\[
\begin{align*}
S0 & = 50; \quad K = 50; \quad M = 200; \quad T = 1; \quad \text{dlt} = T/M; \quad r = 0.05; \quad s = 0.1; \\
rd & = ( r - s^2/2 ) \times \text{dlt}; \quad N = 10000; \quad Z = \text{randn(M,N)}; \\
S & = S0 \times \exp(\text{cumsum}(rd + s \times \text{sqrt(dlt)} \times Z)); \quad \% \text{simple MC} \\
X & = \exp(-r \times T) \times \max(\text{mean}(S)-K, 0); \\
\text{disp}([\text{mean}(X) \quad \text{std}(X) \quad 2 \times \text{std}(X)/\sqrt{N}]) \\
& \quad 1.8269 \quad 2.1632 \quad 0.043264 \\
d & = ( \log(S0/K) + (r+s^2/6) \times T/2 )/(s \times \text{sqrt}(T/3)); \\
SP & = [S0 \times \text{ones}(1,N); \quad S]^{1/(M+1)}; \quad Y = \exp(-r \times T) \times \max(\text{prod}(SP)-K,0); \\
Xb & = \text{mean}(X); \quad Yb = \text{mean}(Y); \quad cs = -\sum((X-Xb) \times (Y-Yb))/\sum((Y-Yb)^2) \\
\mu Y & = \exp(-r \times T) \times \left( S0 \times \exp((r-s^2/6) \times T/2) \times \text{normcdf}(d) \right. \\
& \quad \left. - K \times \text{normcdf}(d-s \times \text{sqrt}(T/3)) \right); \\
C & = X + cs \times (Y - \mu Y); \\
\text{disp}([\text{mean}(C) \quad \text{std}(C) \quad 2 \times \text{std}(C)/\sqrt{N}]) \quad \% \text{control variate MC} \\
& \quad 1.8303 \quad 0.030711 \quad 0.00061421
\end{align*}
\]