ANTITHETIC VARIABLES, CONTROL VARIATES

Variance Reduction Background: the simulation error for some parameter $\theta \approx \bar{X}$, depends on $MSE = Var(\bar{X}) = Var(X)/n$, so the simulation can be more efficient if $Var(X)$ is reduced.

Antithetic Variables

- Key idea: if $X_1$ and $X_2$ are id RVs with mean $\theta$,
  \[ Var\left(\frac{X_1 + X_2}{2}\right) = \frac{1}{4}(Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)), \]
  so variance is reduced if $X_1$ and $X_2$ have $Cov(X_1, X_2) \leq 0$.

- For many simulations, a $\theta$ estimator is $X_1 = h(U_1, \ldots, U_n)$ for some $h$; so consider the antithetic estimator $X_2 = h(1 - U_1, \ldots, 1 - U_n)$. Combined estimator is $(X_1 + X_2)/2$.

Notes:

a) simplest example; if $X_1 = U$, then $X_2 = 1 - U$, $\frac{X_1 + X_2}{2} = \frac{1}{2}$, and $Var(\frac{X_1 + X_2}{2}) = 0$ (perfect negative correlation).

b) Theorem: if $h(X_1, \ldots, X_n)$ is monotone for each variable, $Cov(h(U_1, \ldots, U_n), h(1 - U_1, \ldots, 1 - U_n)) \leq 0$.

c) if $h(Y)$ is monotone, and $Y$ is are not iid uniform, but $Y_i = F_i^{-1}(U_i), \ i = 1, \ldots, n$,

then $X(U) = h(F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n))$ is monotone in $U$s, so use

\[ W = (X(U) + X(1 - U))/2. \]
ANTITHETIC VARIABLES CONTINUED

Antithetic Variable Examples

1. Estimate the integral \( V = \int_0^\infty \ln(1 + x^2)e^{-x} \, dx \);
   Use \( t = 1 - e^{-x} \):

Matlab for MC method with antithetic variates:

\[
N = 1000; \ g = @(t)\log(1+\log(1-t)).^2; \\
T = \text{rand}(1,N); \ X = g(T); \ % \text{simple MC} \\
disp([\text{mean}(X) \ \text{std}(X) \ 2*\text{std}(X)/\sqrt{N}]) \\
0.66848 \quad 0.73278 \quad 0.046345 \\
T = \text{rand}(1,N/2); \ X = (g(T) + g(1-T))/2; \\
disp([\text{mean}(X) \ \text{std}(X) \ 2*\text{std}(X)/\sqrt{N/2}]) \\
0.67881 \quad 0.30933 \quad 0.027667
\]

Could also use \( x = t/(1-t) \), with \( dx = dt/(1-t)^2 \).

Matlab for MC method with antithetic variates:

\[
f = @(x)\log(1+x.^2).*\exp(-x); \\
h = @(t)f(t./(1-t))./(1-t).^2; \\
T = \text{rand}(1,N); \ X = h(T); \ % \text{simple MC} \\
disp([\text{mean}(X) \ \text{std}(X) \ 2*\text{std}(X)/\sqrt{N}]) \\
0.6695 \quad 0.71206 \quad 0.045034 \\
T = \text{rand}(1,N/2); \ X = (h(T) + h(1-T))/2; \\
disp([\text{mean}(X) \ \text{std}(X) \ 2*\text{std}(X)/\sqrt{N/2}]) \\
0.67492 \quad 0.46416 \quad 0.041516
\]
ANTITHETIC VARIABLE EXAMPLES CONT.

2. Estimate $V = \int_0^{\pi/4} \int_0^{\pi/4} x^2y^2 \sin(x + y) \ln(x + y) \, dx \, dy$; notice integrand is monotone increasing in $x$ and $y$.
Use $x = \pi u_1/4$, $y = \pi u_2/4$:

Matlab for MC method with antithetic variates:

```matlab
N = 1000; U = rand(2,N);
f = @(x)prod(x).^2.*sin(sum(x)).*log(sum(x));
X = pi^2*f(pi*U/4)/16; % simple MC
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
    0.0037452  0.012534  0.00079271
U = rand(2,N/2);
X = pi^2*( ( f(pi*U/4) + f(pi*(1-U)/4) )/2 )/16;
disp([mean(X) std(X) 2*std(X)/sqrt(N/2)])
    0.0037885  0.0085332  0.00076323
```
3. Simulating a queueing system with $N$ arrivals.

Two-servers in parallel example.

Inter-arrival times for $N$ arrivals are $-\ln(1 - U_i)/\lambda$, and server times are $G_1^{-1}(V_i)$ or $G_2^{-1}(V_i)$ for $i = 1, \ldots, N$, with $U_i, V_i \sim \text{Uniform}(0, 1)$.

Output (e.g. average time in system) is a function $D(U_1, \ldots, U_N, V_1, \ldots, V_N)$, which should be monotone increasing in $U$s and $V$s.

Matlab results for 2-server parallel queue, ave. time in system:

```matlab
N = 1000; M = 100;
for i=1:M
    U = rand(1,N); V = rand(1,N);
    D(i) = prllsvp(N,U,V,6,4,3);
end
disp([mean(D) std(D) 2*std(D)/sqrt(M)])\% simple MC
0.99208 0.36936 0.073871
for i=1:M/2
    U = rand(1,N); V = rand(1,N);
    A(i)= ( prllsvp(N,U,V,6,4,3) + ... 
            prllsvp(N,1-U,1-V,6,4,3) )/2;
end
disp( [mean(A) std(A) 2*std(A)/sqrt(M/2)])
1.0254 0.20094 0.056835
```
CONTROL VARIATES

Control Variates

• Background: assume desired simulation quantity is $\theta = E[X]$; there is another simulation RV $Y$ with known $\mu_Y = E[Y]$.

For any $c$, RV $Z = X + c(Y - \mu_Y)$, is an unbiased estimator of $\theta$, because $E[Z] = E[X] + c(E[Y] - \mu_Y) = \theta$. Now

$$Var(Z) = Var(X+cY) = Var(X) + c^2Var(Y) + 2cCov(X,Y)$$

is minimized when $c = c^* = -\frac{Cov(X,Y)}{Var(Y)}$, and

$$Var(X + c^*Y) = Var(X) - \frac{Cov(X,Y)^2}{Var(Y)}.$$  \(Y\) is called a control variate for $X$; in order to reduce variance, choose a $Y$ correlated with $X$.

• Implementation: $Cov(X,Y), Var(Y)$ estimated from data.

$$Cov(X,Y) \approx \hat{Cov}(X,Y) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}),$$

$$Var(Y) \approx \hat{Var}(Y) = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2,$$

so

$$c^* \approx \hat{c}^* = -\frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2},$$

• Note: goal is to choose $Y$ so that $Y$ is $\approx X$, with $Y$ is easy to simulate and $\mu_Y$ is easy to find.
CONTROL VARIATES CONTINUED

Control Variate Examples

1. Simple example: to estimate $\theta = \int_0^1 e^x \, dx$, try $Y = U$.
   Then $\mu_Y = 1/2$, $\text{Var}(U) = 1/12$,
   
   $\text{Var}(e^U) = \int_0^1 e^{2x} \, dx - \left( \int_0^1 e^x \, dx \right)^2$
   
   $= (e^2 - 1)/2 - (e - 1)^2 \approx .242$;
   
   $\text{Cov}(e^U, U) = \int_0^1 xe^x \, dx - \int_0^1 x \, dx \int_0^1 e^x \, dx$
   
   $= 1 - (e - 1)/2 \approx .141086$
   
   $\text{Var}(X + c^*Y) = \text{Var}(X) - \text{Cov}(X, Y)^2/\text{Var}(Y)$
   
   $\approx .0039$

Matlab for MC method with control variates:

```matlab
N = 1000; U = rand(1,N); X = exp(U);
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
   1.7488  0.49766  0.031475
Y = U; muY = 1/2;
Xb = mean(X); Yb = mean(Y);
   cs = -sum((X-Xb).*(Y-Yb))/sum((Y-Yb).^2);
Z = X + cs*( Y - muY );
disp([mean(Z) std(Z) 2*std(Z)/sqrt(N)])
   1.7189  0.063119  0.003992
```
CONTROL VARIATES CONTINUED

2. Estimate \( V = \int_0^2 e^{-x^2} \, dx = 2 \int_0^1 e^{-(2u)^2} \, du \).

Try \( Y = 2e^{-2U} \):

Matlab for MC method with control variates:

```matlab
N = 1000; U = rand(1,N);
X = 2*exp(-(2*U).^2);
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
    0.91666  0.68842  0.04354
Y = 2*exp(-2*U); muY = 1 - exp(-2);
Xb = mean(X); Yb = mean(Y);
cs = -sum( (X-Xb).*(Y-Yb) )/sum( (Y-Yb).^2 );
Z = X + cs*( Y - muY );
disp([mean(Z) std(Z) 2*std(Z)/sqrt(N)])
    0.88474  0.13402  0.0084762
```
3. Estimate $V = \int_0^{\pi/4} \int_0^{\pi/4} x^2 y^2 \sin(x + y) \ln(x + y) \, dx \, dy$; Use $x^2 y^2$ to “approximate” the integrand.

Matlab for MC method with control variates:

```matlab
N = 1000; U = rand(2,N);
f = @(x)prod(x).^2.*sin(sum(x)).*log(sum(x));
X = pi^2*f(pi*U/4)/16; Xb = mean(X); % simple MC
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
     0.0035267  0.011023  0.00069718

g = prod(x).^2;
Y = pi^2*g(pi*U/4)/16; Yb = mean(Y);
cs = -sum( (X-Xb).*(Y-Yb) )/sum( (Y-Yb).^2 );
muY = (pi/4)^6/9; Z = X + cs*( Y - muY );
disp([mean(Z) std(Z) 2*std(Z)/sqrt(N)])
     0.0037289  0.0043002  0.00027197
```
CONTROL VARIATE EXAMPLES CONT.

4. Asian option: this has $S_m = S_{m-1}e^{(r-\frac{\sigma^2}{2})\delta + \sigma \sqrt{\delta}Z}$, with $\delta = T/M$, $Z \sim \text{Normal}(0,1)$ and expected profit

$$E[e^{-rT} \max\left(\frac{1}{M} \sum_{i=1}^{M} S_i(Z) - K, 0\right)].$$

Control variate is $Y = e^{-rT} \max\left(\prod_{i=0}^{M} S_i(Z))^{\frac{1}{M+1}} - K, 0\right),$ with $\mu_Y = e^{-rT}(e^{(r-\frac{\sigma^2}{6})T/2}S_0\Phi(d) - K\Phi(d - \sigma\sqrt{\frac{T}{3}})),$

$$d = (\ln(S_0/K) + (r + \sigma^2/6)T/2)/(\sigma\sqrt{T/3}).$$

Typical error is reduced by factor $\approx 50.$

Matlab for MC method with control variates:

```matlab
S0 = 50; K = 50; M = 16; T = 1; dlt = T/M;
r = 0.05; s = 0.1; rd = ( r - s^2/2 )*dlt;
N = 10000; Z = randn(16,N);
S = S0*exp(cumsum(rd + s*sqrt(dlt)*Z));
X = exp(-r*T)*max( mean(S)-K, 0 );
disp([mean(X) std(X) 2*std(X)/sqrt(N)])
1.9522 2.2445 0.044889
Y = exp(-r*T)*max((S0*prod(S)).^(1/(M+1))-K,0);
Xb = mean(X); Yb = mean(Y);
cs = -sum((X-Xb).*Yb)/sum((Y-Yb).^2);
muY = exp(-r*T)*(S0*exp((r-s^2/6)*T/2)*normcdf(d) ... -K*normcdf(d-s*sqrt(T/3)));
Z = X + cs*( Y - muY );
disp([mean(Z) std(Z) 2*std(Z)/sqrt(N)])
1.9375 0.03389 0.0006778
```