PROBABILITY REVIEW

Notation, Basic Probability

- *Sample spaces* S with *events* A_i, *probabilities* $P(A_i)$;
  union $A \cup B$ and *intersection* AB, *complement* A^c.
- Axioms: $P(A) \leq 1$; $P(S) = 1$; for exclusive A_i, $P(\cup_i A_i) = \sum_i P(A_i)$.
- Conditional probability: $P(A|B) = P(AB)/P(B)$;
  $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$
- *Random variables* (RVs) X;
  the *cumulative distribution function* (cdf) $F(x) = P\{X \leq x\}$;
  for a discrete RV, *probability mass function* (pmf)
  $$f(x) = P\{X = x\}, x = x_1, x_2, \ldots; \qquad F(x) = \sum_{x_i \leq x} f(x_i);$$
  for a continuous RV, *probability density function* (pdf)
  $$f(x), \text{ with } P\{X \in C\} = \int_C f(x)dx; \; F(x) = \int_{-\infty}^{x} f(t)dt.$$

PROBABILITY REVIEW CONTINUED

Notation, Basic Probability Continued

- Generalizations for more than one variable, e.g.
  two RVs X and Y: joint cdf $F(x,y) = P\{X \le x, Y \le y\}$;
  pmf $f(x,y) = P\{X = x, Y = y\}$; or
  pdf $f(x,y)$, with $P\{(X,Y) \in A\} = \int\int_A f(x,y)dxdy$;
  independent X and Y iff $f(x,y) = f_X(x)f_Y(y)$.
- *Expected value* or *mean*: for RV X, $\mu = E[X]$; discrete RVs:
  $$E[X] = \sum_i x_i f(x_i), \text{ or } E[g(X)] = \sum_i g(x_i) f(x_i);$$

  continuous RVs: $E[X] = \int_{-\infty}^{\infty} x f(x)dx$, or $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$;

  $E[aX + b] = aE[X] + b = a\mu + b$.

PROBABILITY REVIEW CONTINUED

- *Variance*: $Var(X) = E[(X-\mu)^2]$, with $Var(X) = E[X^2] - \mu^2$, $Var(aX+b) = a^2 Var(X)$, and *standard deviation* $\sigma = \sqrt{Var(X)}$;

  with RVs X, Y, *covariance* $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$, and
  $Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$;
  independent RVs have $Cov(X,Y) = 0$;
  the *correlation* $Corr(X,Y) = Cov(X,Y)/\sqrt{Var(X)Var(Y)}$.

Chebyshev's Inequality : for RV X with μ and σ, $P\{|X-\mu| \geq k\sigma\} \leq 1/k^2$.

Weak Law of Large Numbers : if $X_1, X_2, \ldots,$ is sequence of independent and identically distributed (iid) RVs with mean μ, then for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left\{\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \epsilon\right\} = 0.$$

PROBABILITY REVIEW CONTINUED

Some Discrete RVs

- *Binomial* RVs: n independent trials, success probability is p.
  If X is number of successes,

  $$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i};$$

  with $E[X] = np$, $Var(X) = np(1-p)$;

  if $n = 1$, X is a *Bernoulli* RV.
- *Poisson* RVs: take values $i = 0, 1, 2, \ldots,$ with $P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$;
  and $E[X] = Var(X) = \lambda$.

For small p, Poisson RV's approximate the number of successes in a large number (n) of trials, with $\lambda \approx np$.

PROBABILITY REVIEW CONTINUED

- *Geometric* RVs: for n independent trials, with success probability p. If X is the number of the first success,

$$P\{X = i\} = (1-p)^{i-1}p;$$

with $E[X] = 1/p$, $Var(X) = (1-p)/p^2$.

- *Negative Binomial* RVs: for independent trials with success probability p. If X is the number of trials for r successes,

$$P\{X = n\} = \binom{n-1}{r-1}(1-p)^{n-r}p^r;$$

with $E[X] = r/p$, $Var(X) = r(1-p)/p^2$.

PROBABILITY REVIEW CONTINUED

Some Continuous RVs

- *Uniform* RVs: an RV X which is uniform on $[a, b]$ has pdf

$$f(x) = \begin{cases} 1/(b-a) & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases},$$

  and cdf $F(x) = (x-a)/(b-a)$; with $E[X] = (b+a)/2$, $E[X^2] = (a^2+b^2+ab)/3$, so $Var(X) = (b-a)^2/12$.

- *Exponential* RVs': pdf is $f(x) = \lambda e^{-\lambda x}$, $0 < x < \infty$,
  cdf is $F(x) = 1 - e^{-\lambda x}$; with $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$.

  Exponential RVs are *memoryless*:
  $P\{X > s+t | X > s\} = P\{X > t\}$ or
  $P\{X > s+t\} = P\{X > s\}P\{X > t\} = e^{-\lambda s}e^{-\lambda t}$.

PROBABILITY REVIEW CONTINUED

Continuous RVs Continued

- *Normal* RVs: pdf $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, $-\infty < x < \infty$, and cdf $F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(t-\mu)^2/(2\sigma^2)} dt = \Phi(\frac{X-\mu}{\sigma})$; with $E[X] = \mu$, $Var(X) = \sigma^2$.

Standardized $Z = (X-\mu)/\sigma$ has pdf $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and cdf

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt;\ \text{with } E[X] = 0,\ Var(X) = 1.$$

Central Limit Theorem: if $X_1, X_2, \ldots,$ is a sequence of iid RVs with finite mean μ and finite variance σ^2, then

$$\lim_{n\to\infty} P\left\{ \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} < x \right\} = \Phi(x).$$

Note: this is often used in the form

$$P\left\{ |\bar{X} - \mu| < x\frac{\sigma}{\sqrt{n}} \right\} = 1 - \alpha \approx 2\Phi(x) - 1,$$

to compute an α-*confidence interval* for $\bar{X} = \sum_{i=1}^{n} X_i/n$.

PROBABILITY REVIEW CONTINUED

Continuous RVs Continued

- *Gamma* RVs: pdf $f(x) = \lambda e^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!}$, $0 < x < \infty$;

  cdf $F(x) = 1 - e^{-\lambda x}\sum_{i=0}^{n-1}\frac{(\lambda x)^i}{i!}$, $E[X] = \frac{n}{\lambda}$, $Var(X) = \frac{n}{\lambda^2}$.

Poisson processes: if $N(t)$ is # events occuring in $[0,t]$ with
$N(0) = 0$; events are independent; $\lim_{h\to 0}\frac{P\{N(h)=1\}}{h} = \lambda$;
$N(s+t) - N(s)$ independent of s; and $\lim_{h\to 0}\frac{P\{N(h)\geq 2\}}{h} = 0$.
Conditions imply $N(t)$ is Poisson RV with mean λt.
If X_i is i^{th} inter-arrival time, X_i's are iid exponential with
$P\{\sum_{i=1}^n X_i < t\} = P\{n \leq N(t)\} = e^{-\lambda t}\sum_{i=n}^{\infty}\frac{(\lambda t)^i}{i!}$

Homogeneous Poisson processes have λ independent of t;
Nonhomogeneous Poisson processes have $\lambda(t)$ (dependent on t).

PROBABILITY REVIEW CONTINUED

Conditional Expectation and Variance

$$E[X|Y=y] = \sum_x P\{X=x, Y=y\}/P\{Y=y\} \text{ for discrete RVs, OR}$$
$$= \int xf(x,y)dx / \int f(x,y)dx \qquad \text{for continuous RVs;}$$

conditional variance formula

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]).$$