MATH 416/516 ASSIGNMENT 5 SOLUTIONS

- Problem 9.18:

  a) Raw simulation uses $Y \sim Normal(1,1)$, $X \sim Normal(Y,4)$, and checks if $X > 1$;
  in terms of standard normals Z the simulation uses $\theta \approx \frac{1}{N}\sum_{i=1}^{N} I(1 + Z_{1,i} + 2Z_{2,i} > 1)$.

  b) Conditioning on Y, $P\{X > 1\} = P\{Y + 2Z > 1\} = P\{Z > (1-Y)/2\}$
  $= 1 - \Phi((1-Y)/2) = 1 - \Phi(-Z/2)$, so $\theta \approx \frac{1}{N}\sum_{i=1}^{N}(1 - \Phi(-Z_i/2))$.

  c) Antithetic variables would replace the Z_i's by $-Z_i$'s and take the average:
  $\theta \approx \frac{1}{N}\sum_{i=1}^{N}(1 - \Phi(-Z_i/2) + 1 - \Phi(Z_i/2))/2$.

  But $1 - \Phi(-Z/2) = \Phi(Z/2)$, so $(1 - \Phi(-Z/2) + 1 - \Phi(Z/2))/2 = 1/2$, and
  therefore antithetics give the exact result $\theta = 1/2$.

  e), f), g) Matlab

```
N = 1000; Z = randn(1,N); Y = 1 + Z; I = Y + 2*randn(1,N) > 1;
disp( [mean(I) std(I) 2*std(I)/sqrt(N)] ) % e)
   0.478      0.49977      0.031608
W = 1 - normcdf(-Z/2);
disp( [mean(W) std(W) 2*std(W)/sqrt(N)] ) % f)
  0.49072     0.17711      0.011201
A = ( W + 1 - normcdf(Z/2) )/2;
disp( [mean(A) std(A) 2*std(A)/sqrt(N)] ) % g)
   0.5   3.1856e-17   2.0148e-18
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  i) Analysis in c) shows $\theta = 1/2$.

- Problem 9.19*:

  a) Raw simulation estimates $E[p(U)]$ by generating U, then $N \sim Poisson(\lambda(U))$,
  and counting proportion of N's > 20.

  b) For conditioning use $P\{\#\text{ Claims} \geq 20|U\} = 1 - P\{\#\text{ Claims} < 20|U\}$
  $= p(U) = 1 - e^{-\lambda(U)}\sum_{i=0}^{19}\lambda(U)^i/i!$, with $\lambda(U) = 15/(U + .5)$.

  The simplest control comes from the conditioning variable, just use $Y = U$;
  then $\mu_Y = \lambda(\mu_U) = 15/(\mu_U + .5) = 15$;
  another control comes from $p(U) \approx 1 - exp(-\lambda(U)) \approx 15/(U + .5)$;

  c) antithetics replace Us by (1-U)s.

- Problem 9.26: the simplest case is $E[X|2]$, where two pair have been drawn, so one card is discarded and one is drawn from the 47 which contains 2 of each kind from the original two pair. This redraw could lead to a full house or the same two pair, so

$$E[X|2] = 8P\{full|2\} + 2P\{2|2\} = 8(4/47) + 2(1 - 4/47) \approx 2.5106.$$

With cases 1 or 0, a pair(P) is drawn (say XX, with YZW) and this is kept with three new cards drawn from 47. The 47 contains X,X, 3 each of Y,Z and W, and 4 each of the 9 other card types. The redraw could give 4, full, 3, 2-pair, or nothing better.

Using $R = 1/\binom{47}{3} = 1/16215 \approx .000061671$,

$P\{4|P\}$ comes from XXO with 45 choices for O, so $P\{4|P\} = 45R \approx .0027752$;

$P\{full|P\}$ comes from XOO (probability $2(3\binom{3}{2} + 9\binom{4}{2})R = 126R$) or
OOO (probability $(3 + 9\binom{4}{3})R = 39R$), so $P\{full|P\} = 165R \approx .010176$;

$P\{3|P\}$ from XYO, or XOY so $P\{3|P\} = 2((9(42) + 36(41))R/2 = 1854R \approx .11434$;

$P\{2P|P\}$ from YYO or YOO, so $P\{2P|P\} = (3\binom{3}{2}42 + 9\binom{4}{2}41)R = 2592R \approx .15985$;

and $P\{P|P\} = 1 - (45 + 165 + 1854 + 2592)R = 1 - 4656R \approx .71286$;

then

$$E[X|1] = 25(45)R + 8(165)R + 3(1854)R + 2(2592)R + 1(1 - 4656R) = 1 + 8535R \approx 1.52636;$$

and

$$E[X|0] = 25(45)R + 8(165)R + 3(1854)R + 2(2592)R = 13191R \approx .81351.$$

- Problem 9.27: using the results from the text and problem 26, but with text numbers (4*800+36*50+13*48*25+13*12*24*8+4*(13*11*9-10)*5+10*1020*4)C =.044976 ≠.0512903 (text),

$$\begin{aligned}\hat{\theta} &= .044976 + .021128(4.24144) + .047539(2.5106) + .130021(1.52636) + .29255(.81351) + .50105X_o \\ &= .69671 + .50105X_o.\end{aligned}$$

  Using $N = 100000$ the raw simulation gave $E[X] \approx .8529$ and $Var[X] \approx 3.2$,
  but the conditional simulation gave $E[\hat{\theta}] \approx .85515$, with $Var[\hat{\theta}] \approx .20$,
  so there was a significant variance reduction with conditional simulation.
  See end of solution key for Matlab functions.

- Problem 9.36*: Gamma(3/2,1) has density $K\sqrt{x}e^{-x}$, with $K = 2/\sqrt{\pi}$, so use
  $E[e^X/(X+1)^2] = \int_0^\infty \frac{e^x}{(x+1)^2}e^{-x}dx = \int_0^\infty \frac{e^x}{K\sqrt{x}(x+1)^2}K\sqrt{x}e^{-x}dx.$
  For simulation, estimate $E[W|X]$ with $W = \frac{\sqrt{\pi}e^X}{2\sqrt{X}(X+1)^2}$ and $X \sim Gamma(\frac{3}{2}, 1)$.

- Problem 9.38: in example 9y, rates were $1/(i+2)$, so use those; with $i = 4$, $1/(i+2) = 1/6$.
  b) Let $E_1 = E[I(S_4 > 62)|S_3 = x] = E[X_4 > 62 - x^*)] = \frac{1}{6}\int_{62-x^*}^\infty e^{-\frac{X_4}{6}}dX_4.$, with $x^* = \min(x, 62)$.
  a) Let $E_2 = E[S_4 I(S_4 > 62)|S_3 = x] = E[(x + X_4)I(X_4 > 62 - x^*)]$
  $= \frac{1}{6}\int_{62-x^*}^\infty (x + X_4)e^{-\frac{X_4}{6}}dX_4 = xE_1 + \frac{1}{6}\int_{62-x^*}^\infty X_4 e^{-\frac{X_4}{6}}dX_4 = xE_1 + (62 - x^*)e^{-\frac{x^*}{6}} + 6E_1.$
  c) Generate $x = S_3$ RV values using $x = S_3 = -\sum_{i=1}^3 \ln(U_i)/(i+2)$, then given x, invert
  $U_4 = \frac{\int_{62-x^*}^{X_4} e^{-\frac{X}{6}}dX}{\int_{62-x^*}^\infty e^{-\frac{X}{6}}dX}$, to get X_4; Average $x + X_4$ results for final expectation.

- Problem 9.41: the conditional cdf for $Z > c$ is $\frac{1}{1-\Phi(c)}\int_c^Z \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = (\Phi(Z) - \Phi(c))/(1 - \Phi(c)) = U.$
  Solving for Z, $Z = \Phi^{-1}(\Phi(c) + (1 - \Phi(c))U) = \Phi^{-1}(U + (1 - U)\Phi(c)))$.
  The conditional cdf for $a < Z < b$ is $\frac{1}{\Phi(b)-\Phi(a)}\int_a^Z \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = (\Phi(Z) - \Phi(a))/(\Phi(b) - \Phi(c)) = U.$
  Solving for Z, $Z = \Phi^{-1}(\Phi(a) + (\Phi(b) - \Phi(a))U)$.

Matlab functions for 9.27

```
function [ XB V E] = vidpkr(N) % raw simulation with N samples
% Problem 9.27
%  Output is mean hand value XB, with variance V, estimated error E.
%     Example: [XB V E] = vidpkr(10000); disp([XB V E])
%
for i = 1 : N, pp = zeros(1,10);
  deck = randperm(52); deal = deck(1:5);
  [cards js] = sort(mod(deal,13)+2); suits = ceil(deal(js)/13);
  [ Xp kcards ksuits ] = payoff(cards,suits); l = length(kcards);
  if l < 5, rdeal = deck(6:10-l); % pick new cards, get new payoff
    rcards = mod(rdeal,13)+2; suits = [ksuits ceil(rdeal/13)];
    [cards js] = sort([kcards rcards]); suits = suits(js);
    Xp = payoff(cards,suits);
  end, X(i) = Xp;
end, XB = mean(X); V = var(X); E = 2*sqrt(V/N);
% end vidpkr
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```
function [ X kcards ksuits ] = payoff( cards, suits ), k = [1:5]; X = 0;
% X is value for hand, kcards and ksuits are kept cards and suits
%   assumes input cards are sorted, with values 2:14, Ace = 14.
  if  min(suits) == max(suits)                   % check for flushes
    if sum(cards) == 60,                 X = 800;                    % RF
    elseif cards(1)+4 == cards(5),        X = 50;                    % SF
    elseif cards(4) == 5 & cards(5) == 14, X = 50;     % Ace low     % SF
    else                                   X = 5;                    % FL
    end                                                % check other hands
  elseif cards(1) == cards(4) | cards(2) == cards(5),     X = 25;    % 4K
  elseif cards(1) == cards(3)
    if cards(4) == cards(5), X = 8; else, k = [1:3]; X = 3; end      % FH,3K
  elseif cards(3) == cards(5)
    if cards(1) == cards(2), X = 8; else, k = [3:5]; X = 3; end      % FH,3K
  elseif cards(2) == cards(4),           k = [2:4]; X = 3;           % 3K
  elseif cards(1) == cards(2) & cards(3) == cards(4), k = [1:4]; X = 2; % 2P
  elseif cards(2) == cards(3) & cards(4) == cards(5), k = [2:5]; X = 2; % 2P
  elseif cards(1) == cards(2) & cards(4) == cards(5)
                                                k = [1 2 4 5]; X = 2; % 2P
  elseif sum(   cards(1)+[0:4] == cards) == 5, X = 4;                 % ST
  elseif sum(cards(1)+[0:3 12] == cards) == 5, X = 4;   % Ace low      % ST
  elseif cards(1) == cards(2), k = [1 2]; if cards(1) > 10, X = 1; end % LP,HP
  elseif cards(2) == cards(3), k = [2 3]; if cards(2) > 10, X = 1; end % LP,HP
  elseif cards(3) == cards(4), k = [3 4]; if cards(3) > 10, X = 1; end % LP,HP
  elseif cards(4) == cards(5), k = [4 5]; if cards(4) > 10, X = 1; end % LP,HP
  else, k = [];
  end, kcards = cards(k); ksuits = suits(k); % Kept cards
%

function [ XB V E] = vidpkrc(N) % conditional simulation with N samples
% Problem 9.27
%  Output is mean hand value XB, with variance V, estimated error E.
%     Example: [XB V E] = vidpkr(10000); disp([XB V E])
%
for i = 1 : N
  while 1, deck = randperm(52); deal = deck(1:5);
    [cards js] = sort(mod(deal,13)+2); suits = ceil(deal(js)/13);
    [ Xp kcards ksuits ] = payoff(cards,suits);
    if length(kcards) == 0, break; end
  end, deal = deck(6:10); % pick new cards and get new payoff
  [cards js] = sort(mod(deal,13)+2); suits = ceil(deal(js)/13);
% .05129+.021128*4.24144+.047539*2.5106+.130021*1.52636+.29255*.81351=.69671
% .044976+.021128*4.24144+.047539*2.5106+.130021*1.52636+.29255*.81351=.69039
  X(i) = .69039 + .50105*payoff(cards,suits);
end
XB = mean(X); V = var(X); E = 2*sqrt(V/N);
% end vidpkrc
```