MATH 416/516 MIDTERM EXAM 1 ANSWERS

You may assume the availability of Uniform(0,1) RVs for all algorithms.

1. (12 pts.) Determine an algorithm for the approximation of the integral $\int_0^\infty \frac{\cos(x)}{1+x^4}dx$ using the Monte Carlo method (hint: use the transformation $x = (1-y)/y$, with $y \in (0,1)$).
   Answer: using $x = (1-y)/y$, $dx = -(1/y^2)dy$, $\int_0^\infty \frac{\cos(x)}{1+x^4}dx = \int_0^1 \frac{\cos(y/(1-y))}{1+((1-y)/y)^4}(1/y^2)dy$.
   Algorithm: given the number of samples N, set $S = 0$;
   for i = 1:N,
   set $X_i = (1-U_i)/U_i$, with $U_i \sim Uniform(0,1)$ $S = S + \frac{\cos(X_i)}{(1+X_i^4)U_i^2}$.
   end
   Output $S/N \approx \int_0^\infty \frac{\cos(x)}{1+x^4}dx$.

2. (12 pts.) Determine an algorithm which uses the inverse transform method to generate a random variable with probability density function **(pdf)** $f(x) = 3x^2e^{-x^3}, \quad 0 \le x < \infty$.
   Answer: first compute $F(x) = \int_0^x 3t^2e^{-t^3}dt = e^{-t^3}\Big|_0^x = 1 - e^{-x^3}$;
   then invert $U = F(X) = 1 - e^{-X^3}$, to find $X = (-\ln(1-U))^{1/3}$.
   Algorithm: generate $U \sim Uniform(0,1)$, and output $X = (-\ln(1-U))^{1/3}$.

3. (16 pts.) Determine an algorithm for the generation of a random variable with probability density function **(pdf)**
$$f(x) = \begin{cases} x^3 & 0 \le x < 1 \\ 1 & 1 \le x < \frac{7}{4} \\ 0 & \text{elsewhere} \end{cases}$$
   Answer: first find the cdf. There are two pieces,
   $F_1(x) = \int_0^x t^2dt = x^4/4$, for $0 \le x < 1$, with $F_1(1) = 1/4$, and
   $F_2(x) = F_1(1) + \int_1^x dt = 1/4 + t\Big|_1^x = 1/4 + x - 1 = x - 3/4, \ x \ge 1$.
   Inverting both pieces, $X = (4U)^{1/4}$ for $U < 1/4$, and $X = U + 3/4$, for $U > 1/4$.
   Algorithm: generate $U \sim Uniform(0,1)$;
   if $U < 1/4$, output $X = (4U)^{1/4}$, otherwise output $X = U + 3/4$.

4. (14 pts.) Taxis arrive at an airport according to a homogeneous Poisson process with rate of 10 per hour, and each taxi has a random number N, of passengers, with $P\{N = j\} = 2(3^{-j})$, for $j = 1, 2, \ldots$ (hint: N is a geometric RV). Determine an algorithm to simulate the arrival times for the taxis and the total number of passengers arriving during an 20 hour day. Your algorithm should specify how the arrival times and numbers of passengers could be calculated.
   Answer: Poisson inter-arrival times with rate 10 are $-\ln(U)/10$, and the
   passenger $\#N$ is a geometric RV with $p = 2/3$, so $N = \lceil \ln(U)/\ln(1/3) \rceil$.
   Algorithm:
   generate $U_1 \sim Uniform(0,1)$ and set $t = -\ln(U_1)/10$, and $n = 0$;
   while $t < 20$, generate $U_1, U_2 \sim Uniform(0,1)$,
   set $n = n + \lceil \ln(U_2)/\ln(1/3) \rceil$ and $t = t - \ln(U_1)/10$;
   end while
   Output total passengers n.

5. (14 pts.) Determine an algorithm which uses the **inverse transform method** to generate a random variable with probability mass function **(pmf)** given by

$$P\{X = k\} = k\ 2^{-(k+1)}, \quad k = 1, 2, \ldots.$$

Answer: notice $p_1 = 1/4$, $p_2 = 2/8$, and $p_j = jp_{j-1}/(2(j-1))$.

The algorithm is the standard algorithm for a discrete distribution:

Algorithm:

1) generate $U \sim Uniform(0,1)$, set $F = 1/4, p = 1/4, j = 1$;
2) while $U > F$, set $j = j+1$, $p = jp/(2(j-1))$, $F = F + p$; end while
3) Output $X = j$.

6. (16 pts.) Determine an algorithm which uses the **composition method** to generate a random variable with probability density function **(pdf)**

$$f(x) = \frac{1}{5} + \frac{3\sqrt{x}}{80}, \quad 0 \le x \le 4$$

Answer: first determine $F(x) - \int_0^x f(t)dt = x/5 \quad + \quad x^{3/2}/40$;

then write F as a weighted sum of two cdf's, for $0 \le x \le 4$.

At $x = 4$, $x^{3/2} = 8$, and $F(4) = 4/5 + 1/5$, so weights must be 4/5 and 1/5, with

$F(x) = (\frac{4}{5})\frac{x}{4} + (\frac{1}{5})\frac{x^{3/2}}{8} = (\frac{4}{5})F_1(x) + (\frac{1}{5})F_2(x)$, with $F_1(x) = x/4,\ F_2(x)x^{3/2}/8$.

Inverting the pieces, $X = 4U$ or $X = 4U^{2/3}$, checking that both map $(0,1)$ to $(0,4)$.

Algorithm: generate $U_1, U_2 \sim Uniform(0,1)$;

if $U_1 < 4/5$, output $X = 4U_2$, otherwise output $X = 4U_2^{2/3}$.

7. (16 pts.) Determine an **acceptance-rejection** algorithm for the generation of a random variable X with probability density function **(pdf)**

$$f(x) = 2\frac{(\sin(x))^2}{\pi}, \quad 0 \le x \le \pi,$$

using $g(x) = 1/\pi$. Also, determine the approximate number of uniform RV's the algorithm will use to generate each X.

Answer: using $h(x) = f(x)/g(x) = 2\sin(x)^2$; the maximum occurs where $\sin(x)$ is max at $x = \pi/2$, with $\max(h) = 2 = c$, the AR constant, and $f/(cg) = \sin(x)^2$;

Algorithm: generate $U_1, U_2 \sim Uniform(0,1)$, and set $X = \pi U_2$;

while $U_1 > \sin(X)^2$, generate $U_1, U_2 \sim Uniform(0,1)$, and set $X = \pi U_2$; end while

Output X.

The expected number of AR steps is $c = 2$, and each AR step uses two U's, so the expected number of U's is 4.