

MATH 416/516 MIDTERM EXAM 2

Show all work and justify your answers.

1. (10 pts.) To estimate $\theta = E[X]$, X_1, X_2, \dots, X_{100} have been simulated with sample variance $S^2 = 1$. Approximately how many more simulation runs would be needed if we want the standard deviation of the estimator for $E[X]$ to be less than .05? In this case, what would an approximate 95% confidence interval for θ be?

2. (12 pts.) Explain in detail how antithetic variables would be used for a simulation to estimate

$$\theta = \int_0^1 \int_0^1 e^x \sin((x^2 + y^3)/2) dx dy.$$

Explain why antithetic variables should reduce variance for this problem, compared to raw simulation.

3. (20 pts.) Suppose Y is a uniform(0,1) random variable, and conditional on $Y = y$, X is a normal random variable with mean y and variance 9.

- Explain in detail how raw simulation would be used to estimate $\theta = P\{X > 2\}$.

- Explain in detail how conditional expectation would be used to obtain an improved estimator for θ .

4. (25 pts.) Assume simulation is used to estimate $\theta = E[X]$ for some output variable X , and another output variable Y is available with $E[Y] = \mu_Y$ known.

- If $W = X + c(Y - \mu_Y)$, show that $E[W] = E[X]$.
- By considering $Var(W) = Var(X + c(Y - \mu_Y))$, show that

$$Var(W) = Var(X) + c^2 Var(Y) + 2c Cov(X, Y).$$

Hint: for RVs X, Y , and constants a, b , you may assume

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab Cov(X, Y).$$

- Show that $Var(W)$ is minimized when $c = c^* = -Cov(X, Y)/Var(Y)$.

- If $\theta = \int_0^1 e^{-x^2} \cos(x) dx$ describe in detail how raw simulation could be used to estimate θ , and how the control variable $Y = \cos(X)$ would be used with simulation to provide a reduced variance estimator for θ .

5. (16 pts.) Explain in detail how stratified sampling with 10 equal-width strata and $n = 10000$ total samples would be used with a simulation for

$$\theta = \int_0^1 e^x \sin(x^2) dx.$$

6. (17 pts.) Explain in detail how raw simulation with exponential (mean 1) random variables would be used to estimate

$$\theta = \int_0^\infty \frac{1}{1 + (x - 1)^2} e^{-x} dx.$$

Explain in detail how importance sampling could be used to provide an estimator for θ with reduced variance (hint: use a tilted density).