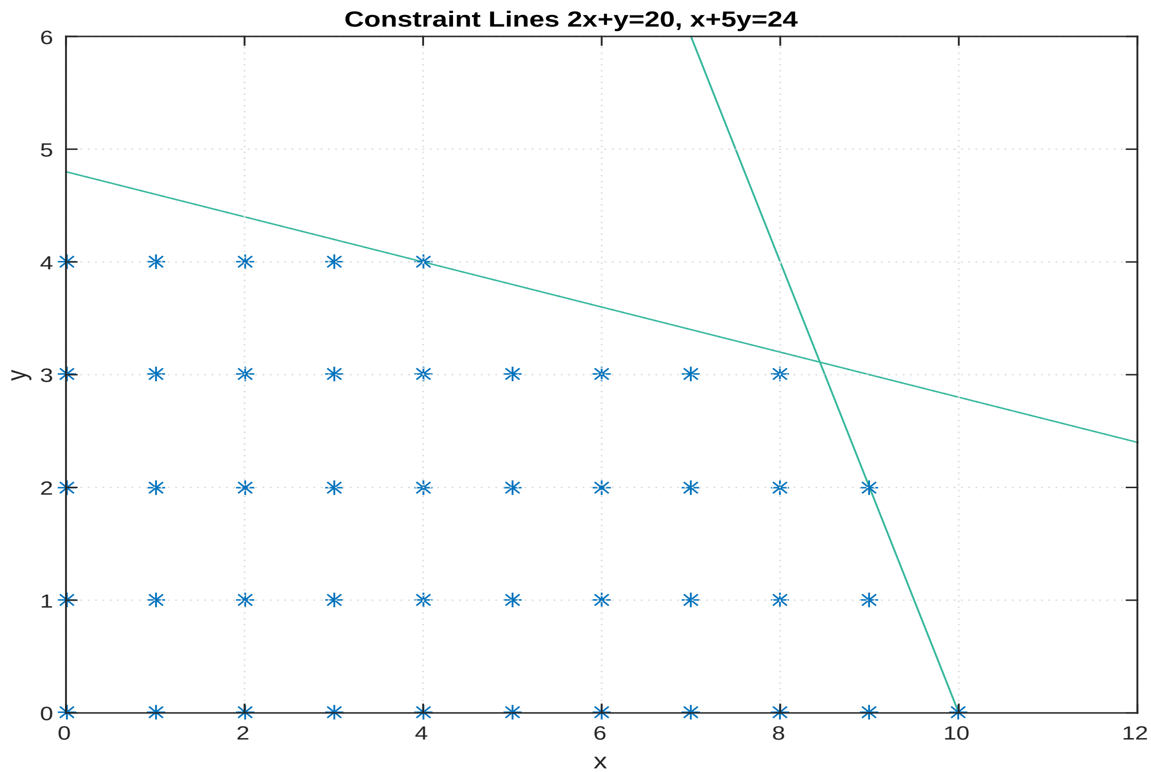
GOMORY'S CUTTING PLANE ALGORITHM

Gomory Algorithm Background :

- Consider standard LP problem with all variables restricted to integers
- Basic strategy:
  a) use simplex algorithm to solve LP;
  b) iteratively add constraints (cutting planes) to find optimal integer solution.
- Cutting Plane Example A (Exer. 6.1.2): Maximize $2x_1 + 9x_2$,
  subject to $2x_1 + x_2 \leq 20$, $x_1 + 5x_5 \leq 24$, with integer $x_1,\ x_2 \geq 0$.

  Simplex solution $(8\frac{4}{9},\ 3\frac{1}{9})$; $z = 44\frac{8}{9}$.



GOMORY'S CUTTING PLANE ALGORITHM CONT.

Cutting Plane Example A: first new constraint

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{5}{9}$	$-\frac{1}{9}$	0	$\frac{76}{9}$
x_2	0	1	$-\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{28}{9}$
x_5	0	0	$-\frac{5}{9}$	$-\frac{8}{9}$	1	$-\frac{4}{9}$
	0	0	$\frac{1}{9}$	$\frac{16}{9}$	0	$\frac{404}{9}$
x_1	1	0	0	-1	1	8
x_2	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{16}{5}$
x_3	0	0	1	$\frac{8}{5}$	$-\frac{9}{5}$	$\frac{4}{5}$
	0	0	0	$\frac{8}{5}$	$\frac{1}{5}$	$\frac{224}{5}$

Page1 of 1 printed on April 14, 2016 by genz at 11:15:43 AM PDT

GOMORY'S CUTTING PLANE ALGORITHM CONT.

Cutting Plane Example A: second new constraint

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	-1	1	0	8
x_2	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{16}{5}$
x_3	0	0	1	$\frac{8}{5}$	$-\frac{9}{5}$	0	$\frac{4}{5}$
x_6	0	0	0	$-\frac{3}{5}$	$-\frac{1}{5}$	1	$-\frac{4}{5}$
	0	0	0	$\frac{8}{5}$	$\frac{1}{5}$	0	$\frac{224}{5}$
x_1	1	0	0	-4	0	5	4
x_2	0	1	0	1	0	-1	4
x_3	0	0	1	7	0	-9	8
x_5	0	0	0	3	1	-5	4
	0	0	0	1	0	1	44

Page1 of 1 printed on April 14, 2016 by genz at 11:16:12 AM PDT

GOMORY'S CUTTING PLANE ALGORITHM CONT.

Gomory Algorithm Details : let $[a]$ be greatest integer $\leq a$ (rounding down), and define the *fractional part of* a to be $= a - [a]$.

1. Begin with LP in standard form for application of simplex method.
2. Apply simplex method until convergence, and select any noninteger b_i^* constraint:
$$\sum_j a_{ij}^* x_j = b_i^*$$
3. Rewrite constraint using fractional parts $f_{ij} = a_{ij} - [a_{ij}]$, $f_i = b_i - [b_i]$:
$$\sum_j f_{ij}^* x_j - f_j^* = [b_j^*] - \sum_j [a_{ij}^*] x_j$$
Heuristic for step 2: choose b_i^* with largest f_i^*.
4. Add new constraint $\sum_j f_{ij} x_j - f_j \geq 0$, with integer excess, to tableau.
5. Repeat steps 2-4 (using dual simplex) until all rhs b_i^*'s are integers.

GOMORY'S CUTTING PLANE ALGORITHM CONT.

Gomory Algorithm Analysis :

- For each iteration, consider converged simplex tableau at step 2 with $m \times n$ A^*:

$$\begin{array}{c|c|c} & \mathbf{x}^t & \\ \hline \mathbf{x}_B^* & A^* & \mathbf{b}^* \\ \hline & \mathbf{c}^{*t} & z_0^* \end{array}.$$

  For any noninteger basic variable $x_{B(i)}$

$$x_{B(i)} = b_i^* = \sum_j a_{i,j}^* x_j \geq \sum_j [a_{i,j}^*] x_j \geq \sum_j [a_{i,j}^*][x_j].$$

- So, for integer $\mathbf{x}$, $\sum_j [a_{i,j}^*] x_j \leq [x_{B(i)}]$; add integer slack $x_{n+1} \geq 0$.

$$[b_i^*] = \sum_j [a_{i,j}^*] x_j + x_{n+1}.$$

  Integer solutions to original problem must satisfy new constraint.

- Subtract to get extra constraint for tableau:

$$f_i^* = \sum_j f_{i,j}^* x_j - x_{n+1}, \quad \text{or} \quad -\sum_j f_{i,j}^* x_j + x_{n+1} = -f_i^*.$$

- New constraint (plane) "cuts" off part of original feasible region, but revised tableau has same integer solutions as original problem.
- Issues with Gomory algorithm: no integer solution until end; rounding errors.

GOMORY'S CUTTING PLANE ALGORITHM CONT.

Cutting Plane Example B: Maximize $5x_1 + 6x_2$, with integer $x_1,\ x_2 \geq 0$.
subject to $10x_1 + 3x_2 \leq 52$, $2x_1 + 3x_2 \leq 18$. Final simplex tableau is

	x_1	x_2	x_3	x_4	$\mathbf{b}^*$
x_1	1	0	1/8	−1/8	17/4
x_2	0	1	−1/12	5/12	19/6
	0	0	1/8	15/8	161/4

Revised final tableau is

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}^*$
x_1	1	0	0	−1	1	4
x_2	0	1	0	1	−2/3	10/3
x_3	0	0	1	7	−8	2
	0	0	0	1	1	40

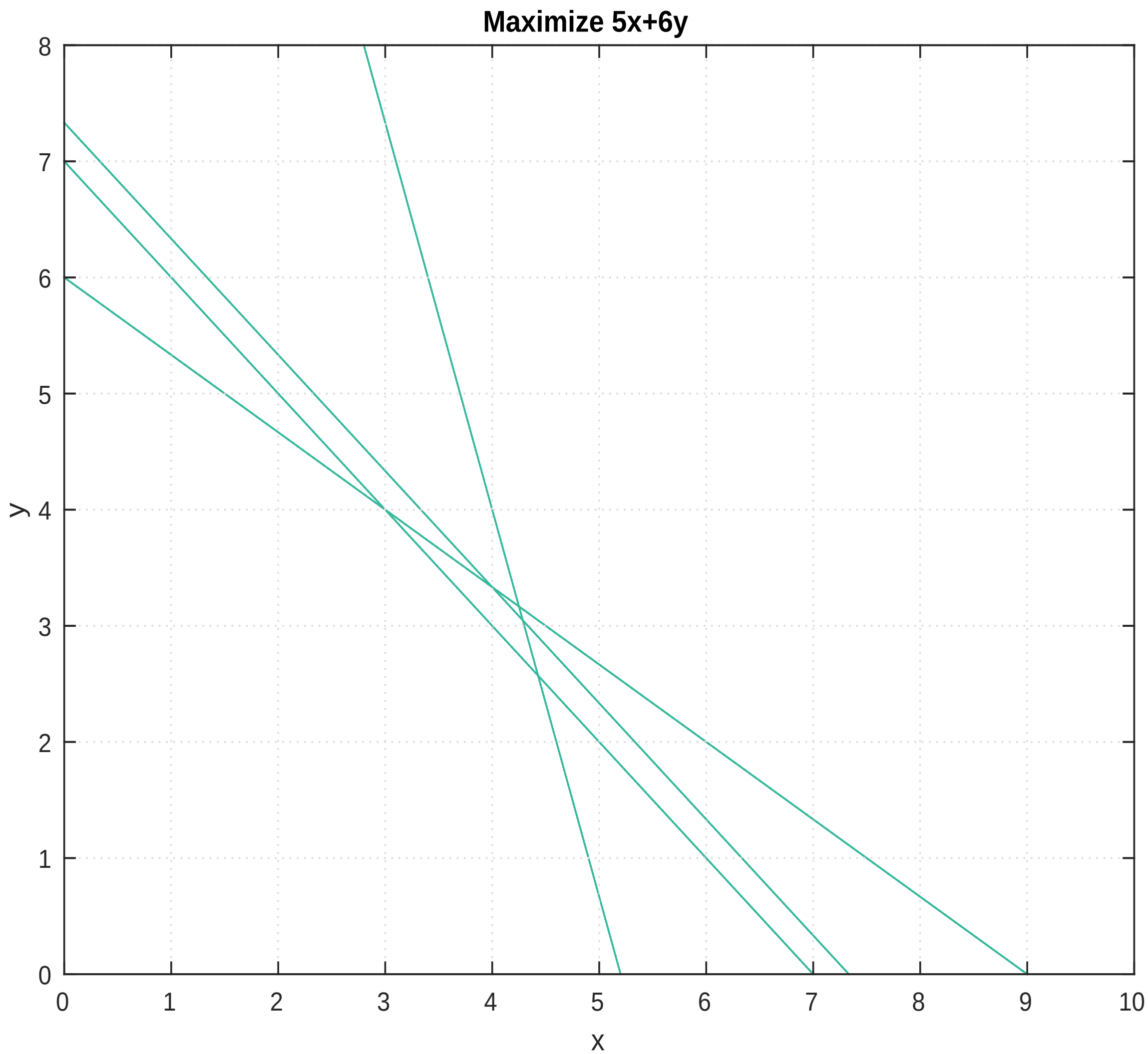
Final final tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}^*$
x_1	1	0	0	−1	0	3	3
x_2	0	1	0	1	0	−2	4
x_3	0	0	1	7	0	−24	10
x_4	0	0	0	0	1	−3	1
	0	0	0	1	0	3	39

GOMORY'S CUTTING PLANE ALGORITHM CONT.

Cutting Plane Example B Graph with extra constraints

$$\{-x_3 - 7x_4 \leq -2, \quad -x_5 + 3x_6 \leq -1\} \;\rightarrow\; \{3x_1 + 3x_2 \leq 22, \quad x_1 + x_2 \leq 7\}.$$



GOMORY'S CUTTING PLANE ALGORITHM CONT.

Gomory Algorithm for Mixed Integer Programming Problems :

- Basic idea: find cutting planes using coefficients for integer variables.
- Algorithm Details
  1. Begin with LP in standard form for application of simplex method.
  2. Apply simplex method until convergence, and select any noninteger b_i^* constraint for an integer basic variable $x_{B(i)}$: $\sum_j a_{ij}^* x_j = b_i^*$;
$$x_{B(i)} + \sum_{NBVs} a_{i,j}^* x_j = [b_i^*] + f_i^*.$$
  3. Divide NBV indices into N^+, with $a_{i,j}^* \geq 0$, and N^- otherwise.
$$\sum_{j \in N^+} a_{i,j}^* x_j + \sum_{j \in N^-} a_{i,j}^* x_j = f_i^* + ([b_i^*] - x_{B(i)}).$$
  4. Two cases, depending $f_i^* + ([b_i^*] - x_{B(i)}) < 0$ or > 0:
  if < 0, with $[b_i^*] - x_{B(i)}$ a negative integer, and $0 < f_i^* < 1$, so
$$\sum_{j \in N^+} a_{i,j}^* x_j + \sum_{j \in N^-} a_{i,j}^* x_j \leq f_i^* - 1, \implies \sum_{j \in N^-} a_{i,j}^* x_j \leq f_i^* - 1;$$
$$\implies \sum_{j \in N^+} a_{i,j}^* x_j + \frac{f_i^*}{f_i^* - 1} \sum_{j \in N^-} a_{i,j}^* x_j \geq f_i^*.$$

GOMORY'S CUTTING PLANE ALGORITHM CONT.

Gomory algorithm for mixed IPs step 4 continued: other case implies same inequality, so add slack x_{n+1}, and new constraint to tableau

$$-\sum_{j\in N^+} a^*_{i,j}x_j - \frac{f^*_i}{f^*_i - 1}\sum_{j\in N^-} a^*_{i,j}x_j + x_{n+1} = -f^*_i;$$

5. Repeat (dual simplex) steps 2-4 until all b^*_i's for integer x_i's are integers.

- Mixed IP Gomory Example: Maximize $5x_1 + 6x_2$, with $x_2 \geq 0$, integer $x_1 \geq 0$, subject to $10x_1 + 3x_2 \leq 52$, $2x_1 + 3x_5 \leq 18$.

Final simplex tableau is

	x_1	x_2	x_3	x_4		$\mathbf{b}^*$
x_1	1	0	1/8	−1/8		17/4
x_2	0	1	−1/12	5/12		19/6
	0	0	1/8	15/8		161/4

Revised final tableau is

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}^*$
x_1	1	0	0	−1/6	1	4
x_2	0	1	0	4/9	−2/3	10/3
x_3	0	0	1	1/3	−8	2
	0	0	0	11/6	1	40