1. (10 pts) Classify the following ODE. Determine the order, whether it is linear or nonlinear, whether it is autonomous, and if linear, whether it is homogeneous. Circle all that are correct. (NO EXPLANATION NECESSARY)

(a) \( \frac{d^2y}{dt^2} + 3ty^3 = 1 \)
A) first-order, second-order, third-order  B) linear, nonlinear
C) homogeneous, non-homogeneous

(b) \( y' + 6y = e^y \)
A) first-order, second-order, third-order  B) linear, nonlinear
C) homogeneous, non-homogeneous  D) autonomous, non-autonomous
2. (10 pts) Find the general solution of the following first order initial value problem:

\[ y' + 2ty = 5t \]

\[ y = e^{t^2} \]

\[ \frac{d}{dt}(ye^{t^2}) = 5te^{t^2} \]

\[ ye^{t^2} = \int 5te^{t^2} \, dt = \frac{5}{2} e^{t^2} + c \]

\[ y = \frac{5}{2} + ce^{-t^2} \]

\[ \int \frac{dy}{x^2 - 2y} = \int \frac{d\sqrt{2y}}{\sqrt{x - 2y}} \]

\[ -\frac{1}{2} \ln |x - 2y| = \frac{1}{2} t^2 + c \]

3. (10 pts) Find the solution of the initial value problem. You may leave your answer in implicit form.

\[ y' = \frac{x}{1 + 2y}, \quad y(-1) = 0 \]

\[ (1 + 2y) \, dy = x \, dx \]

\[ y + y^2 = \frac{1}{2} x^2 + c \]

\[ y(-1) = 0 \]

\[ 0 + 0 = \frac{1}{2} + c \quad \Rightarrow \quad c = -\frac{1}{2} \]

\[ y + y^2 = \frac{1}{2} x^2 - \frac{1}{2} \]
4. (10 pts) Find the general (implicit) solution of the differential equation:

\[
\frac{dy}{dx} = \frac{3x^2 + y}{3y^2 - x}
\]

**Solution**

This is a nonlinear equation, so Integrating Factor won’t work. Doesn’t look separable, so let’s see if it is an exact equation. Rearranging we get

\[ Pdx + Qdy = (-3x^2 - y)dx + (3y^2 - x)dy = 0 \]

Check: \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \)? And we get -1=-1 so it is exact : ).

So our goal is to now find \( F(x, y) \) such that

\[
\frac{\partial F}{\partial x} = P = -3x^2 - y \quad \frac{\partial F}{\partial y} = Q = 3y^2 - x.
\]

Then the solution will be \( F(x, y) = C \).

Begin with the first equation:

\[
\frac{\partial F}{\partial x} = -3x^2 - y \\
F(x, y) = -x^3 - xy + \phi(y)
\]

Now we use the second equation to determine \( \phi(y) \):

\[
\frac{\partial F}{\partial y} = \frac{\partial (-x^3 - xy + \phi(y))}{\partial y} = -x + \phi'(y) = 3y^2 - x = Q \\
\phi'(y) = 3y^2 \\
\phi(y) = y^3
\]

so we get

\[
F(x, y) = -x^3 - xy + y^3.
\]

Thus our solution is:

\[-x^3 - xy + y^3 = C \quad \text{or} \quad x^3 + xy - y^3 = C \]

(both are correct since \( C \) is an arbitrary constant).
5. (10 pts) A 100 gallon tank initially contains 50 gallons of pure water. Water containing a salt concentration of 3 grams per gallon flows into the tank at a rate of 3 gallons per minute. The salt solution in the tank is pumped out at the rate of 2 gallons per minute. What is the initial value problem that describes the amount of salt in the tank at any time \( t \) until the tank overflows? Assume that the solution in the tank is kept perfectly mixed at all times. **DO NOT SOLVE THIS INITIAL VALUE PROBLEM.**

**Solution**

Define all variables carefully: Let

- \( x \) mass of salt in tank (grams)
- \( t \) time (minutes), \( t = 0 \) when flow in/out starts
- \( V \) volume of fluid in tank (gallons)

Using: \( \frac{dx}{dt} = \text{rate in} - \text{rate out} \), checking units as we go, we get

\[
\frac{dx}{dt} = 3 \left( \frac{x}{V(t)} \right) - 2 \frac{x}{\text{gall} \text{ grams min}} - \frac{\text{gall grams min}}{\text{gall}}
\]

The volume of fluid in the tank is **not** constant as there is more coming in than going out. At \( t = 0 \), \( V = 50 \), at \( t = 1 \), \( V = 51 \), and at \( t = 2 \), \( V = 52 \), so using this pattern we get \( V(t) = 50 + t \). We can also use \( \frac{dV}{dt} = 3 - 2 \) and \( V(0) = 50 \) to obtain the same result:

\[
V(t) = 50 + t
\]

Substituting in we obtain the differential equation:

\[
\frac{dx}{dt} = 9 - \frac{2x}{50 + t}
\]

To make this an **initial value problem** we need an initial condition. Using the statement that the tank initially contains 50 gallons of **pure water** (implies no salt), we have:

\[
x(0) = 0
\]
6. (4 pts) True or False (circle one): Based on existence and uniqueness theorems (no explanation required).

(a) True  False
The solution to \( y' + t^2 y = \sin t \) with an initial condition is guaranteed to exist and be unique for all values of \( t \).
Remark: This is a linear equation so the solution will exist in the entire rectangle of our choosing that satisfy the existence and uniqueness theorem. In this case we can choose \( R \) to be of infinite width: \(-\infty < t < \infty\). (The height of the rectangle can also be made arbitrarily large, so any initial condition will be inside this \( R \)).

(b) True  False
The solution to \( y' = y^{1/3} \), \( y(0) = 0 \) is guaranteed to exist and be unique for all values of \( t \).
Remark: This is a nonlinear equation so we are guaranteed a unique solution only near the initial condition.

7. (10 pts) For differential equation

\[
\frac{dy}{dt} = (1 - y)(y - 2).
\]

(a) Determine the equilibrium (critical) points for the equation.

(b) Determine whether the equilibrium solutions are asymptotically stable or unstable.
6. (4 pts) True or False (circle one): Based on existence and uniqueness theorems (no explanation required).

(a) True   False
The solution to \( y' + t^2 y = \sin t \) with an initial condition is guaranteed to exist and be unique for all values of \( t \).

(b) True   False
The solution to \( y' = y^{1/3}, y(0) = 0 \) is guaranteed to exist and be unique for all values of \( t \).

7. (10 pts) For differential equation

\[
\frac{dy}{dt} = (1 - y)(y - 2).
\]

(a) Determine the equilibrium (critical) points for the equation.

\[
(1 - y)(y - 2) = 0 \quad \Rightarrow \quad y = 1, y = 2
\]

\( y = 1 \) and \( y = 2 \) are two equilibrium solutions.

(b) Determine whether the equilibrium solutions are asymptotically stable or unstable.

\[
f(y) = (1 - y)(y - 2) = -y^2 + 3y - 2
\]

\[
f'(y) = -2y + 3
\]

\[
f'(1) = -2 \cdot 1 + 3 = 1 > 0 \quad \Rightarrow \quad y = 1 \text{ unstable}
\]

\[
f'(2) = -2 \cdot 2 + 3 = -1 < 0 \quad \Rightarrow \quad y = 2 \text{ asymptotically stable.}
\]
8. (10 pts) Find the general solution (in real form) for the equation $y'' + 2y' + 2y = 0$.

Characteristic Eq. $r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

General solution: $y = e^{-t} (c_1 \cos t + c_2 \sin t)$.

9. (10 pts) Find the solution to the initial value problem: $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$.

Characteristic Eq. $r^2 + 3r + 2 = 0$.

$$(r+1)(r+2) = 0$$

$r_1 = -1$, $r_2 = -2$.

General solution: $y = c_1 e^{-t} + c_2 e^{-2t}$.

$y(0) = c_1 + c_2 = 1$ \quad $\Rightarrow$ \quad $-c_1 = 0$ \quad $\Rightarrow$ \quad $c_2 = -1$

$y'(0) = -c_1 - 2c_2 = -1$ \quad $\Rightarrow$ \quad $c_1 = 1$

$\Rightarrow$ \quad $y(t) = e^{-t}$. 

$\Rightarrow$