Homework 2 Solution Key

Assigned problems: 11.6 # 19, 27, 30, 36, 70
11.7 # 16, 19, 46, 48
11.8 # 24, 33

11.6 # 19

Derivative of \( r(t) = 2 + 4t + 6t^{3/2} + \frac{10}{3}t \) at \( t = 1 \)

First, we will rewrite \( r(t) \) to have a term similar to that of the other 2 components of the vector, giving us \( +10t^{-1} \):

\[
\begin{align*}
  \vec{r}'(t) &= \left( 2 + 4t \right) \hat{i} + \left( 6t^{3/2} \right) \hat{j} + \left( \frac{10}{3}t \right) \hat{k} \\
  &= 2(1) \hat{i} + 6(1^{3/2}) \hat{j} + \left( \frac{10}{3} \right) \hat{k} \\
  &= 2(1) \hat{i} + 6(1) \hat{j} + \left( \frac{10}{3} \right) \hat{k} \\
  &= \left( \frac{8}{3}, 9, \frac{10}{3} \right)
\end{align*}
\]

To find \( \vec{r}'(1) \), all we need to do is plug in \( 1 \) in \( \vec{r}(t) \) in \( \vec{r}'(t) \):

\[
\begin{align*}
  \vec{r}'(1) &= \left( \frac{8}{3}, 9, \frac{10}{3} \right) \\
  \vec{r}(1) &= \left( \frac{8}{3}, 9, \frac{10}{3} \right)
\end{align*}
\]

# 27 Unit tangent vector of \( \cos(2t), 9, 3 \sin(2t) \) for \( 0 \leq t \leq \pi, \ t = \frac{\pi}{2} \)

Since we want the unit tangent vector, we first need to find a tangent vector, which means that we need to find \( \vec{r}'(t) \):

\[
\begin{align*}
  \vec{r}'(t) &= \left( \cos(2t), 9, 3 \sin(2t) \right) \\
  \vec{r}(t) &= \left( \cos(2t), 9, 3 \sin(2t) \right) \\
  &= \left( \cos(2t), 9, 3 \sin(2t) \right) \quad \text{(Chain rule)} \\
  &= \left( -2 \sin(2t), 0, 6 \cos(2t) \right)
\end{align*}
\]

To find the tangent vector at \( t = \frac{\pi}{2} \), we need to plug \( \frac{\pi}{2} \) in \( \vec{r}'(t) \):
\[ r(t) = <-2\sin(2t), 0, 6\cos(2t)> \]
\[ r\left(\frac{\pi}{2}\right) = <-2\sin(\pi), 0, 6\cos(\pi)> = <-2, 0, -6> \]

Now we have a tangent vector. To make it a unit tangent, we need to divide the vector by its magnitude.

\[ |r\left(\frac{\pi}{2}\right)| = \sqrt{(-2)^2 + 0^2 + (-6)^2} = \sqrt{36} = 6 \]

Unit vector: 
\[ \frac{r\left(\frac{\pi}{2}\right)}{|r\left(\frac{\pi}{2}\right)|} = \frac{<-2, 0, -6>}{6} = \frac{1}{6}<-2, 0, -6> \]

\[ \text{Integral integral of } r(t) = \int e^{t^2} + \sin(1 - t^2) \, dt \]

To find the integral of \( r(t) \), we need to find the integral of all the individual components. That is,

\[ \int e^{t^2} \, dt, \int \sin(1 - t^2) \, dt, \int 2t \, dt \]

We will now proceed to take each integral separately.

\[ \int e^{t^2} \, dt: \quad u = t, \quad du = e^t \]
\[ du \, t = \frac{1}{2}u \, du = e^t \]

\[ \int u \, du = \frac{1}{2}u^2 \]

\[ = \frac{1}{2}e^{2t} + C \]

\[ \int \sin(1 - t^2) \, dt \]

\[ = \sin(1 - t^2) - C \]

\[ = \sin(1 - t^2) + C \]

\[ \int 2t \, dt = t^2 + C \]

\[ = (t - 1)e^t + C \]
\[ S(t \cdot \sin(t^2))\, dt : \ y = t^2, \quad du = 2\,dt, \quad \int \sqrt{2} \, dt = \sqrt{2} \, t + C_2 \]

\[ S(\frac{1}{\sqrt{1 - y^2}})\, dy : \ y = \sin(u), \quad du = \cos(u)\, du \]

\[ = \frac{1}{2} \int \cos(u) \, du \]

\[ = \frac{1}{2} \sin(u) + C_3 \]

\[ = \frac{1}{2} \sin(1) \cos^2(u) + C_3 \]

\[ S(-2\,t)\, dt : \ y = \frac{1}{2} + t, \quad du = 2\,dt \]

\[ S(-2\,t \cdot t^2)\, dt = -\frac{1}{2} t^2 \, du \]

\[ = -t^3 + C_3 \]

\[ = -t^3 + C_3 \]

Thus,
\[ S(t\,dt) = \langle 5\,t \cdot 0\, dt, \ 4\,t \cdot \sin(t^2)\, dt, \ \sqrt{2} \cdot \cos(t^2)\, dt \rangle \]

\[ = \langle 4\,t \cdot 0\, dt, \ 4\,t \cdot \sin(t^2)\, dt, \ \sqrt{2} \cdot \cos(t^2)\, dt \rangle \]

Taking the constant terms out gives
\[ S(t\,dt) = \langle -\frac{1}{2} t^3 + C_3, \ -t^3 + C_3, \ -t^3 + C_3 \rangle \]

#56: Find \( r(t) \) satisfying \( \dot{r}(t) < 16, r(t) \cdot 1^2, r(1) = \langle 2, 3, 4 \rangle \)

Since we have \( r(t) \) and we want \( r(t) \), we need to take the integral of \( r(t) \)

\[ r(t) = \int r(t)\, dt = \int \langle 5\,t \cdot 0\, dt, \ 4\,t \cdot \sin(t^2)\, dt, \ \sqrt{2} \cdot \cos(t^2)\, dt \rangle \]

\[ = \int \langle 5\,t \cdot 0\, dt, \ 4\,t \cdot \sin(t^2)\, dt, \ \sqrt{2} \cdot \cos(t^2)\, dt \rangle \]

We will now proceed to evaluate the 3 integrals individually

\[ \int 5\,t^{1/2} \, dt \]

\[ = \frac{2}{3} t^{3/2} + C \]
\[ \sin(t) \, dt: \quad y = \sin \theta \cdot \theta \, dt \rightarrow \theta = \sin^{-1} y \\ \cos(t) \, dt = \frac{1}{2} \cos(\sin^{-1} y) \, dy \\ = \frac{1}{2} \sin(\sin^{-1} y) + C_1 \\ = \frac{1}{2} y + C_2 \]

\[ \sin(\theta) \, d\theta = \sin\theta \cdot \theta \, d\theta \rightarrow \theta = \sin^{-1} \theta \\ \cos(\theta) \, d\theta = \frac{1}{2} \cos(\sin^{-1} \theta) \, d\theta \\ = \frac{1}{2} \sin(\sin^{-1} \theta) + C_3 \\ = \frac{1}{2} \theta + C_4 \]

Therefore, we have that \[ r(t) = \left< \frac{e^{1/2}}{2} + \frac{3}{2}, \sin(\pi t)/\pi, y(\ln t) \right> + \left< 2, 2, 0 \right> \\
We must now solve for the c vector. Knowing that \[ r(1) = \left< 2, 2, 0 \right> \text{, we have} \]
\[ r(1) = \left< 3, 4 \right> = \left< \frac{e^{1/2}}{2} + \frac{3}{2}, \sin(\pi t)/\pi, y(\ln t) \right> + \left< 2, 2, 0 \right> \\
\frac{e^{1/2}}{2} + \frac{3}{2} = 2 \\
\sin(\pi t)/\pi = 2 \\
y(\ln t) + 2 = 0 \]

Thus, \[ r(t) = \left< \frac{e^{1/2}}{2} + \frac{3}{2}, \sin(\pi t)/\pi, y(\ln t) \right> + \left< 2, 2, 0 \right> \\
= \left< \frac{e^{1/2}}{2} + \frac{3}{2}, \sin(\pi t)/\pi + 3, y(\ln t) + 4 \right> \]

#70 Tangent line at \( (\sqrt{2t+1}, \sin \pi t, 4) \) at \( t = 0 \)

For the equation of the tangent line, we need \( r'(t) \) and \( r(t) \) (Similar to point and slope from algebra). 

\[ r(t) = \sqrt{2t+1}, \sin(\pi t), y \rightarrow \left< 3, 0, 9 \right> \]

\[ r'(t) = \left( \sqrt{2t+1}, \sin(\pi t) \right) \rightarrow \left< 0, \pi \cos(\pi t) \right> \]

\[ \left( \sqrt{2t+1} \right)' = \frac{1}{2} \left( 2t + 1 \right)^{-1/2} \cdot 2 \\
= \frac{1}{2} \left( 2t + 1 \right)^{-1/2} \\
= \frac{1}{\sqrt{2t+1}} \]
\[
\begin{align*}
\sin^2(\pi t) &= \cos(\pi t) \cdot (\pi t) \\
&= \pi \cos(\pi t)
\end{align*}
\]

\( y = 0 \)

\[
\begin{align*}
\mathbf{r}'(t) &= \langle \pi \cos(\pi t), \pi \sin(\pi t), 0 \rangle \\
\mathbf{r}(t) &= \langle \frac{\pi}{2}, \pi \sin(\pi t), 0 \rangle \\
&= \langle \frac{\pi}{2}, \pi, 0 \rangle
\end{align*}
\]

Therefore, using (4) and (7), the tangent line is
\[
\mathbf{r}(t) = \langle 3, 0, 4 \rangle + \lambda \langle \frac{\pi}{2}, \pi, 0 \rangle
\]

11.7 #10 Velocity, speed, and acceleration given \( \mathbf{r}(t) = \langle 1 - t^2, 3 + 2t^3 \rangle \)

The velocity is the first derivative of position, thus is \( \mathbf{r}'(t) \)
\[
\begin{align*}
\mathbf{r}'(t) &= \langle -(2t), 6t^2 \rangle \\
\end{align*}
\]

The speed is the magnitude of the velocity, thus is \( |\mathbf{r}'(t)| \)
\[
|\mathbf{r}'(t)| = \sqrt{(-2t)^2 + (6t^2)^2} \\
= \sqrt{4t^2 + 36t^4} \\
= \sqrt{4t^2(1+9t^2)} \\
= 2t \sqrt{1+9t^2}
\]

The acceleration is the velocity's derivative, or 2nd derivative of position
\[
\mathbf{r}''(t) = (\mathbf{r}'(t))' = \langle -2, 12t \rangle
\]
Find velocity, speed, and acceleration for \( r(t) = (2e^{2t} + 1, e^{2t} - 1, 2e^{2t}) \).

The velocity is the derivative of position, thus is \( v(t) \):

\[
v(t) = (\frac{d}{dt}(2e^{2t} + 1), \frac{d}{dt}(e^{2t} - 1), \frac{d}{dt}(2e^{2t})) = (4e^{2t}, 2e^{2t}, 4e^{2t})
\]

The speed is the magnitude of velocity, thus is \( |v(t)| \):

\[
|v(t)| = \sqrt{(4e^{2t})^2 + (2e^{2t})^2 + (4e^{2t})^2} = \sqrt{16e^{4t} + 4e^{4t} + 16e^{4t}} = \sqrt{36e^{4t}} = 6e^{2t}
\]

The acceleration is the velocity's derivative, or 2nd derivative of position:

\[
a(t) = (\frac{d}{dt}(4e^{2t}), \frac{d}{dt}(2e^{2t}), \frac{d}{dt}(4e^{2t})) = (8e^{2t}, 4e^{2t}, 8e^{2t})
\]

Exercise 46: Given \( a(t) = (t, e^t, 1) \), initial velocity \( v(0) = (0, 0, 1) \) and initial position \( r(0) = (4, 0, 2) \).

We know \( v(t) \), and since the integral of acceleration is velocity, we have:

\[
v(t) = \int a(t) dt = \int (t, e^t, 1) dt = (\frac{t^2}{2} + C_x, e^t + C_y, t + C_z)
\]

Furthermore, we know initial velocity \( v(0) \) to be \( (0, 0, 1) \). Thus:

\[
v(0) = (0, e^0, 1) = (0, 1, 1) = (0 + C_x, e^0 + C_y, 0 + C_z)
\]

From each component, we have \( 0 = 0 + C_x \), \( e^0 = 1 + C_y \), and \( 1 = 0 + C_z \). Thus,

\[
z = 0, y = 0, \text{ and } v(t) = (\frac{t^2}{2}, 1, t + 1)
\]
The surge general idea can be expressed to get position, velocity's integral
\[ p(t) = 5 v(t) dt = <x, y, z>, \]
\[ = <x, y, z> + e^{-t} <x, y, z> + e^{-t} <x, y, z> <x, y, z> 
\]
We know the initial position to be \( <4, 0, 0> \) (\( r(0) \)). Thus,
\[ r(0) = <4, 0, 0> = <0, 0, 0> + e^{-0} <x, y, z> + e^{-0} <x, y, z> \]
By each individual component, we have \( y = 0 \), \( x = 0 \), \( 0 = 0 \). Thus,
\[ r(0) = <4, -1, 0>, \text{ and } r(2) = <73, 9, 12> + e^{-2} <x, y, z> \]

A ball is hit east with speed \( <50, 0, 30> \) m/s. A crossword clues the ball south at an acceleration of \( -3 \) m/s. Let \( x \) be east, \( y \) north, and \( z \) be vertical.

- Find velocity and position.

We are told that the wind pushes the ball south at \( -3 \) m/s. For humidity, due to gravity, the ball falls (negative z) at \( -9.8 \) m/s. Thus, our acceleration is \( <0, -3, -9.8> \). The velocity is the integral of acceleration.

\[ v(t) = 50 dt = <0, -3t, -9.8t> + <0, 0, 0> \]

Therefore, \( c = 50, c_x = 0 \) and \( c_z = 90 \). Using this, we have \( v(t) = <50, -3t, -9.8t> \).

The integral of velocity is position, thus the position in this case is
\[ p(t) = 50 dt = <50t, -3t^2/2, -9.8t^2/2> + <0, 0, 0> \]

The ball was hit from the origin, so \( p(0) = <0, 0, 0> \). Thus,
\[ p(0) = <0, 0, 0> = <50t, -3t^2/2, -9.8t^2/2> + <0, 0, 0> \]

Therefore, \( c = 0 \) and our position is \( <50t, -3t^2/2, -9.8t^2/2> \).
b: We will save (b) for the end of this question as the answers to (c) and (d) will help us sketch a graph.

c: Determine time at flight and range of object:
The ball hits the ground when the \( z \) component of the position vector is 0, so \(-9.8t + 30t^2 = 0\). Factoring out \( t \), we have \( t(30 - 9.8t) = 0\). Thus, \( t = 0 \) (when ball is hit) or \( 30 - 9.8t = 0 \) (when the ball hits the ground). Solving for \( t \) gives \( t = \frac{30}{9.8} \approx 3.06 \text{ sec} \).

To find the range, we will find how much the ball moves in the \( x \) and \( y \) directions:
\[
x: 50.6 \cdot 3.06 = 156 \text{ m}
\]
\[
y: -9.8(3.06)^2 = -93 \text{ m}
\]

By the Pythagorean Theorem, the range of the ball is
\[
\text{Range} = \sqrt{(50.6 \text{ m})^2 + (-93 \text{ m})^2} = \sqrt{9306.05 \text{ m}^2} = 96.5 \text{ m}
\]

d: Maximum height of the ball:
The ball is at its maximum height when the upward velocity (\( z \) component of \( v(t) \)) is 0. Thus, \(-9.8t + 30 = 0\). Solving for \( t \) gives \( t = \frac{30}{9.8} \approx 3.06 \text{ sec} \). Alternatively, the ball is at maximum height halfway in its flight time, so \( \frac{6.12 \text{ sec}}{2} = 3.06 \text{ sec} \). To find the height, calculate the \( z \) component of \( p(t) \) for \( t = 3.06 \text{ sec} \:
\[
-9.8t^2 + 30t = -9.8(3.06)^2 + 30(3.06) \approx 93 \text{ m}
\]

b: Now that we have a few points, we can graph the curve. The ball was hit at \((0, 0, 0)\) and landed at \((3.06, -93, 0)\). It is not \((0, 0, 2)\). The ball was at maximum height at \( z = 93 \text{ m} \).
11.8 #20  Speed and length for \( f(t) = \langle 5 \cos(4t), 5 \sin(12t) \rangle \) \( 0 \leq t \leq 2 \)

Speed is the magnitude of the velocity. Since we know the position, we can take the derivative of position to get velocity:

\[ v(t) = f'(t) = \langle -20 \sin(4t), 60 \cos(12t) \rangle \]

The speed is the magnitude of the velocity, or in this case:

\[ s(t) = \sqrt{(-20 \sin(4t))^2 + (60 \cos(12t))^2} = \sqrt{400 \sin^2(4t) + 3600 \cos^2(12t)} = \sqrt{400 \sin^2(4t) + 3600 \cos^2(12t)} = \sqrt{400 + 3600} = \sqrt{4000} = 20 \sqrt{10} \]

The length is the integral of speed evaluated at the endpoints:

\[ \int_0^2 \sqrt{4000} \, dt = 13 \sqrt{10} \approx 52 \]

#33 Length of the spiral \( r = \theta \), for \( 0 \leq \theta \leq 2 \pi \)

Length of a polar curve can be found by \( \int_{\phi_1}^{\phi_2} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta \). In this case, we have:

\[ \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta \]

Let \( u = \theta + y \), then \( du = d\theta \), so \( du \theta = \theta d\theta \), so we have:

\[ \frac{3}{2} \int_0^{2\pi} \sqrt{(\theta^2 + 4)^2 - \frac{1}{3}} \, d\theta = \frac{1}{3} (\sqrt{4} + 1) - \frac{1}{3} (\sqrt{4} + 1) = \frac{1}{3} (\sqrt{4} + 1) = \frac{5}{3} \]

\[ \approx 1.8 = \pi \approx 3.14 \]

\[ \approx 0.929 \]