Certain lacunary cosine series are recurrent

by

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Abstract. Let the coefficients of a lacunary cosine series be bounded and not square-summable. Then the partial sums of the series are recurrent.

In this note, we wish to consider the partial sums of a lacunary cosine series

$$(1) \quad s_N(x) = \sum_{j=1}^{N} a_j \cos(n_j x + \theta_j)$$

where we assume that $a_j, \theta_j$ are real, $|a_j| \leq 1$ for every $j$, and the $n_j$ satisfy $n_{j+1}/n_j \geq \lambda > 1$. We will show:

THEOREM. Suppose that in addition to the conditions just described, also $\sum |a_j|^2 = \infty$. Then $\{s_N(x)\}$ is dense in $\mathbb{R}$ for almost every $x \in [0, 2\pi]$.

We will restate the conclusion in probabilistic terms by saying that for almost all $x$ the sequence $s_N(x)$ is recurrent in $\mathbb{R}$. In the case when $a_j = 1$ for all $j$, this question was posed by T. Murai [4] (see also Brannan and Hayman [3]), and was solved by D. Ullrich [5]. Ullrich was unable to obtain the more general case when $|a_j| \leq 1$; however, the proof we give for our theorem, although shorter than that of Ullrich, does take its key idea from his method.

When the terms $\cos(n_j x + \theta_j)$ in (1) are replaced by $\exp(in_j x)$, the most general conditions which give recurrence of the partial sums in the complex plane are far from being known. Anderson and Pitt [2] showed recurrence in this case when $a_j = 1$ and $n_j = b^j$ for some integer $b$. They have also shown recurrence in the complex plane when $|a_j| \to 0$ or if the $a_j$ are bounded and $n_{j+1}/n_j \to \infty$ [1]. We stress that even though the series we consider are

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the real parts of the series considered by Anderson and Pitt, our results and techniques seem to give little insight into the complex case.

We borrow a lemma from [5]; since its proof is short, we reproduce it here for completeness.

**Lemma.** Suppose that two sequences of sets $E_N, F_N \subset [0, 2\pi]$ have the following property: There exists a constant $c > 0$ and a sequence $\delta_N > 0$ converging to zero such that for every $x \in E_N$ there is an interval $I$ of length $\delta_N$ containing $x$ with $|F_N \cap I| \geq c|I|$. Suppose that for almost every $x \in [0, 2\pi], x \in E_N$ infinitely often. Then for almost every $x \in [0, 2\pi], x \in F_N$ infinitely often.

**Proof.** If we suppose the contrary, then there exists a set $A$ with $|A| > 0$ and a $K$ such that $A \cap (\bigcup_{n=1}^{\infty} F_n)$ is empty. Almost all points of $A$ are points of density, so we can pick a point which is both a point of density of $A$ and in infinitely many $E_N$. But then, for any interval $I$ that contains $x$ and $N > K$, $|I \cap A| \leq |I \cap F_N|$, so that as $|I| \to 0$, $|I \cap F_N|/|I| \to 1$, which contradicts our hypothesis.

**Proof of Theorem.** Let $a \in \mathbb{R}$ and $\varepsilon > 0$ be given. By Zygmund [6] (Vol. 1, p. 205) we have

$$\sup_N s_N(x) = +\infty \quad \text{and} \quad \inf_N s_N(x) = -\infty$$

for almost every $x$. Because of this, the set

$$E_N = \{x \in [0, 2\pi] : s_N(x) \geq a \text{ and } s_{N+1}(x) < a\}$$

covers almost every $x \in [0, 2\pi]$ infinitely often. We establish the conditions of the lemma with the sets

$$F_N = \{x \in [0, 2\pi] : |s_N(x) - a| < \varepsilon \text{ or } |s_{N+1}(x) - a| < \varepsilon\}.$$

Notice that

$$\|s_N'\|_{\infty} \leq \sum_{j=1}^{N} n_j \leq \frac{\lambda}{\lambda - 1} n_N.$$

Let $x \in E_N$. Let $I$ be an interval of length $\delta_N = 2\pi/n_{N+1}$ with center $x$. Since $s_N(x) > a$, there are two cases.

**Case I:** $s_N(x_0) = a$ for some $x_0 \in I$. In this case, because of the estimate for the derivative of $s_N$, there is an interval of length $(\lambda - 1)\varepsilon/(\lambda n_N) > c|I|$ around $x_0$ where $|s_N - a| < \varepsilon$ and $c$ is a constant depending only on $\lambda$. At least half of this interval is in $I \cap F_N$.

**Case II:** $s_N > a$ on $I$. Since $x \in E_N$, $s_N(x) > a$ and $s_{N+1}(x) < a$. Since $I$ is long enough to contain a complete period of $\cos(n_{N+1}x + \theta_{N+1})$, there
is \( x_1 \in I \) such that \( \cos(n_{N+1}x_1 + \theta_{N+1}) = 0 \), giving
\[
s_{N+1}(x_1) = s_N(x_1) + a_{N+1} \cos(n_{N+1}x_1 + \theta_{N+1}) = s_N(x_1) > a.
\]
Thus \( s_{N+1} - a \) has a zero in \( I \) and the argument above gives an interval of length \( (\lambda - 1)\varepsilon/(\lambda n_{N+1}) > c|I| \), centered at this zero, where \( |s_{N+1} - a| < \varepsilon \).
So \( |I \cap F_N| > \frac{\varepsilon}{2}|I| \).

Applying the lemma, we see that for almost all \( x \), \( |s_N(x) - a| < \varepsilon \) infinitely often. Since \( a \in \mathbb{R} \) and \( \varepsilon > 0 \) are arbitrary, the result is proved.

We would like to thank David Ullrich for bringing his paper to our attention.

References

[2] ——, ——, On recurrence properties of certain lacunary series, II. The series \( \sum_n \exp(ian) \), ibid., 83–96.