Integer Optimization (Spring 2009) — Homework 3 Solutions

1. Notice that a set represented by linear inequalities using only continuous variables is a polyhedron. A union of polyhedra (disjunction) could be represented using continuous and 0–1 variables if they have the same recession cone. If the recession cones are not the same for a set of polyhedra, then that set is not b-MIP-r. Analyzing the sets given here for $n = 1$ or $n = 2$ will provide a good idea of which case holds.

   (a)
   $$S = \{\boldsymbol{y} \in \mathbb{R}^n \mid \sum_{i=1}^{n} |y_i| \leq 1\}.$$

   For $n = 1$, $|y_1| \leq 1$ can be represented by the inequalities $y_1 \leq 1, \; -y_1 \leq 1$. For any n, the set S can be represented by the collection of 2^n inequalities (with each y_i taking either $+$ or $-$ sign) as follows:

   $$S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^{n} \pm y_i \leq 1\}.$$

   If we use the continuous variables v_i to represent $|y_i|$, the set in question can be represented using only $2n + 1$ inequalities. Since the main inequality bounds the sum of $|y_i|$, we can use $v_i \geq |y_i|$, which is represented by $v_i \geq y_i \wedge v_i \geq -y_i$. Thus, we get the following representation, which appears more "compact" than the first one.

   $$S = \{\boldsymbol{y} \in \mathbb{R}^n, \boldsymbol{v} \in \mathbb{R}^n_{\geq 0} \mid v_i \geq y_i, v_i \geq -y_i, \sum_{i=1}^{n} v_i \leq 1\}.$$

   The motivation to try and represent the set without adding extra variables is to potentially keep the "size" (number of variables and constraints) of the representation small. As illustrated by this instance, one could reduce the number of constraints significantly by adding a few more variables.

   Of course, if a set can be represented using *only* continuous variables, there is *no need* to use extra 0–1 variables.

   (b)
   $$S = \{\boldsymbol{y} \in \mathbb{R}^n \mid \sum_{i=1}^{n} |y_i| \geq 1\}.$$

   The set S is not a polyhedron. For instance, when $n = 1$, the set is given by $y_i \in (-\infty, -1] \cup [1, \infty)$. Thus it cannot be represented using only continuous variables (options (a) and (b) are ruled out). In fact, S is not b-MIP-r either, as the recession cones of the polyhedra are not the same. For $n = 1$, the recession cones are $(-\infty, 0]$ and $[0, \infty)$. Thus, option (c) is also ruled out in this case.

(c)

$$S = \{\boldsymbol{y} \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \geq 1, \; -M \leq y_i \leq M \; \forall i\}.$$

The addition of bounds on y_i makes the polyhedra bounded. S is still not a single polyhedron, and hence cannot be represented using continuous variables alone (for $n = 1$, S is given by $[-M, -1] \cup [1, M]$). But the recession cones of all the polyhedra are identical (all of them consist of just the origin, as all the polyhedra involved are *polytopes*). Hence, the set is b-MIP-r. We introduce continuous variables v_i, represent $v_i \leq |y_i|$, and then add $\sum_{i=1}^n v_i \geq 1$ to the bounds. $v_i \leq |y_i|$ is equivalent to $v_i \leq y_i \vee v_i \leq -y_i$. Choosing $z_i \in \{0, 1\}$, we write for each i

$$\begin{array}{rcl} v_i & \leq & -y_i + 2Mz_i, \\ v_i & \leq & y_i + 2M(1 - z_i), \\ 0 \leq v_i & \leq & M. \end{array} \tag{1.1}$$

We get the following representation.

$$S = \{\boldsymbol{y} \in \mathbb{R}^n, \boldsymbol{v} \in \mathbb{R}^n, \boldsymbol{z} \in \{0,1\}^n \mid -M \leq y_i \leq M, \; \sum_{i=1}^n v_i \geq 1, \text{ and } (1.1) \; \forall \, i\}.$$

(d)

$$S = \{\boldsymbol{y} \in \mathbb{R}^n \mid \sum_{i=1}^n c_i |y_i| \leq 1\}.$$

If all $c_i \geq 0$, then the set in question is similar to the first set considered (can be represented using 2^n inequalities in the original variables, or using $2n+1$ inequalities using n additional continuous variables). But if c_i could be both positive and negative, the set is generally not representable. For $n = 1$, with $c_1 = -1$, we get the second set considered earlier. Again, for $n = 2$, with $c_1 = 1, c_2 = -1$, we get disjoint polyhedra with non-identical recession cones.

(e)

$$S = \{\boldsymbol{y} \in \mathbb{R}^n \mid \sum_{i=1}^n c_i |y_i| \leq 1, \; -M \leq y_i \leq M \; \forall i\}.$$

The treatment of this set is similar to that of the third set considered. This set is not a polyhedron, but is b-MIP-r (as all the recession cones consist of just the origin). We model $v_i \leq |y_i|$ as we did in for the third set, and add $\sum_{i=1}^n c_i v_i \leq 1$.

(f)

$$S = \{\boldsymbol{y} \in \mathbb{Z}^n \mid A\boldsymbol{y} \leq \boldsymbol{b}, \; \boldsymbol{y} \neq \boldsymbol{y}^*, \; \boldsymbol{y}^* \in \mathbb{Z}^n \; \text{ is fixed}\}.$$

Note that $\boldsymbol{y} \in \mathbb{Z}^n$ here (as opposed to being in $\mathbb{R}^n$). Thus, $\boldsymbol{y} \neq \boldsymbol{y}^*$ can be represented by formulating $|y_i - y_i^*| > 0 \; \forall \; i$, or equivalently, $|y_i - y_i^*| \geq 1 \; \forall \; i$. Use $v_i = y_i - y_i^*$, and then use the result used in representing the third set.

2. One way to prove that P is a sharp formulation of S is to show that every corner point of P is integral. Any corner point of P must be feasible, i.e., satisfy *all* $2n+1$ inequalities ($2n$ bounds on x_i's and $e^T x \geq p$), and must also satisfy (at least) n of the inequalities as equations. If all n of them are bounds, then the corner point is integral. Suppose that one of the corner points of P satisfies $e^T x = p$ and $n-1$ bounds as equations. This vertex is also integral, as p and n are integers. Notice that $n-1$ tight bounds and the main equation forces *all* n variables to satisfy tight bounds.

3. We are considering the set

$$S = \{(x, y_1, y_2, y_3) \in \{0,1\}^4 \mid x = 1 \Rightarrow \text{ at least two of the } y_i\text{'s are } 1\},$$

and the following inequalities.

$$\begin{aligned} x &\leq y_1 + y_2. && (3.2)\\ x &\leq y_1 + y_3. && (3.3)\\ x &\leq y_2 + y_3. && (3.4)\\ 2x &\leq y_1 + y_2 + y_3. && (3.5)\\ 0 &\leq x \leq 1. && (3.6)\\ 0 &\leq y_i \leq 1 \ (i = 1,2,3). && (3.7) \end{aligned}$$

The four formulations we want to compare are

- Formulation 1: bounds ((3.6),(3.7)) and (3.5);
- Formulation 2: bounds ((3.6),(3.7)), (3.5), and (3.2);
- Formulation 3: bounds ((3.6),(3.7)) (3.5), (3.2), and (3.3); and
- Formulation 4: all constraints listed (including bounds).

(a) Constraint (3.5) is the representation of the implication describing S. Adding the bounds, we see that Formulation 1 is valid for S. The constraints (3.2), (3.3), and (3.4) are all valid for S, and hence Formulations 2,3,4 are also valid.

(b) A corner point of Formulation 1 is $(\frac{1}{2}, 0, 0, 1)$, which satisfies three bounds (for the y_i's) and the constraint (3.5) as equations.

(c) By adding constraint (3.2), we strengthen Formulation 1. For instance, the point $(\frac{1}{2}, 0, 0, 1)$ in Formulation 1 violates this constraint.

(d) The point $(\frac{1}{2}, 0, 1, 0)$ in Formulation 2 violates constraint (3.3) in Formulation 3.

(e) The point $(\frac{1}{2}, 1, 0, 0)$ in Formulation 3 violates constraint (3.4) in Formulation 4.

(f) We show that every corner point of Formulation 4 is integral. Note that each corner point has to satisfy four of the constraints as equations (constraints (3.2), (3.3), (3.4), (3.5), and the eight bounds). Consider the following cases.

i. Constraints (3.2), (3.3), (3.4), and (3.5) are tight:
The solution for this set of four equations is $x = y_1 = y_2 = y_3 = 0$, which is an integral corner point.

ii. Constraints (3.2), (3.3), (3.4), and one independent bound are tight:
Equations (3.2), (3.3), and (3.4) give $x = y_1 = y_2 = y_3 = 0$.

iii. Constraints (3.2), (3.3), (3.5), and one independent bound are tight:
Equations (3.2), (3.3), and (3.5) imply $y_1 = 0,\ x = y_2 = y_3$. Hence the bound has to be on one of x, y_2, or y_3, in which case the other two bounds are also rendered tight, thus giving an integral corner point.

iv. Constraints (3.2), (3.3), and two independent bounds are tight:
These equations give $y_2 = y_3,\ x = y_1 + y_2 = y_1 + y_3$. The two independent bounds could be on $(x, y_1),\ (x, y_2),\ (x, y_3),\ (y_1, y_2)$, or (y_1, y_3). As long as the bounds are feasible (with the rest of the inequalities), the corner point obtained will have integral coordinates in each case.

v. Constraints (3.2), (3.5), and two independent bounds are tight:
Equations (3.2) and (3.5) give $x = y_1 + y_2 = y_3$. The two independent bounds could be on $(x, y_1),\ (x, y_2),\ (y_1, y_2),\ (y_1, y_3)$, or (y_2, y_3), with the case of $y_1 = y_2 = 1$ being ruled out because it violates (for instance) $x \le 1$. In each case here, all the variables in the corner point take integral values.

vi. Constraints (3.2) and three independent bounds are tight:
The three independent bounds could be on $(x, y_1, y_3),\ (x, y_2, y_3)$, or (y_1, y_2, y_3). In each case, the fourth variable will be forced to one of its bounds too.

vii. Constraints (3.5) and three independent bounds are tight:
The three independent bounds could be on any three out of the four variables. In all cases, the fourth variable is also forced to one of its bounds to give an integral corner point.

viii. Four bounds are tight:
If all the variables are at bounds, then either on of the constraints (3.2), (3.3), (3.4), and (3.5) will also be tight, or will be violated. Hence, this case is ruled out.