Integer Optimization (Spring 2009) — Homework 1 Solutions

1. For $\lambda \in [0,1]$,
$$\begin{aligned} f(\,\lambda x + (1-\lambda) y\,) &= \textstyle\sum_{i=1}^m f_i(\,\lambda x + (1-\lambda)y\,) \\ &\le \textstyle\sum_{i=1}^m \lambda f_i(x) + (1-\lambda) f_i(y), \quad (\text{ as } f_i'\text{'s are convex}) \\ &= \lambda\,(\textstyle\sum_{i=1}^m f_i(x)) + (1-\lambda)\,(\textstyle\sum_{i=1}^m f_i(y)) \\ &= \lambda f(x) + (1-\lambda) f(y), \text{ hence } f \text{ is convex.} \end{aligned}$$

2. Recall that the unit circle centered at the origin under the infinity norm will actually be the square centered at the origin, with edges that are two units long (the corner points $(1,1), (-1,-1), (-1,1)$, and $(1,-1)$ all have infinity norm of 1, so do the points $(1,0), (0,1)$, and $(-1, 0.35)$, for instance). The goal here is to find the largest circle that fits inside the given polytope P when using the Euclidean norm, and to find the largest *square* that fits inside P when using the infinity norm. Our goal is to find the center $\boldsymbol{y}$ of the circle of radius r (in infinity norm), such that all the points in the circle are also inside the polytope $P = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{a}_i^T \boldsymbol{x} \le \boldsymbol{b} q_i,\ i = 1, \ldots, m\}$ (we assume the system $A\boldsymbol{x} \le \boldsymbol{b}$ consists of m inequalities). To this end, we note that

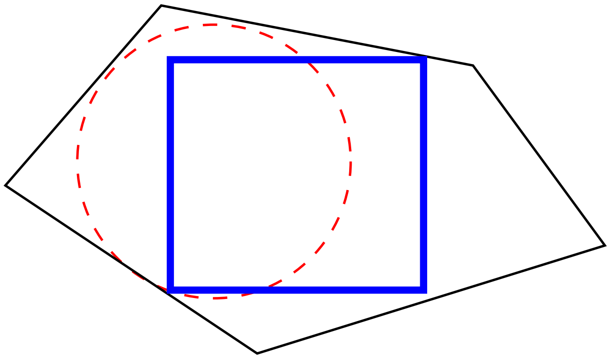

Figure 1: Closed polygon represents P. Dotted circle is the Chebychev circle using Euclidean norm, and the square is the same using infinity-norm.

$$\boldsymbol{x} = \boldsymbol{y} + r \frac{\boldsymbol{a}_i}{\|\,\boldsymbol{a}_i\,\|}$$

is the point in the circle of infinity radius r centered at $\boldsymbol{y}$ that gives the maximum value of $\boldsymbol{a}_i^T \boldsymbol{x}$. Here, $\|\,\boldsymbol{a}_i\,\|$ is the length (Euclidean norm) of $\boldsymbol{a}_i$, and $\boldsymbol{a}_i / \|\,\boldsymbol{a}_i\,\|$ is the unit vector along $\boldsymbol{a}_i$. The idea is the same as the one involved in maximizing $\boldsymbol{c}^T \boldsymbol{x}$ in an LP – if we move the objective function line in the direction of the vector $\boldsymbol{c}$, we increase the value of $\boldsymbol{c}^T \boldsymbol{x}$.

Thus, we want this point $\boldsymbol{x}$ to satisfy the constraints $A\boldsymbol{x} \le \boldsymbol{b}$. In other words, we need

$$\boldsymbol{a}_i^T (\,\boldsymbol{y} + r\,\boldsymbol{a}_i / \,\|\,\boldsymbol{a}_i\,\|\,) = \boldsymbol{a}_i^T \boldsymbol{y} + r\,\|\,\boldsymbol{a}_i\,\| \quad \le \boldsymbol{b}_i, \quad i = 1, \ldots, m.$$

These constraints also imply $A\boldsymbol{y} \le \boldsymbol{b}$, as $r \ge 0$. Maximizing r subject to the above constraints will give the conventional Chebychev center (using Euclidean norm). Similarly, to model the infinity norm, we can write the following LP.

$$\begin{array}{llcll} \max & r & & & \\ \text{s.t.} & \boldsymbol{a}_i^T \boldsymbol{y} + r\,\|\,\boldsymbol{a}_i\,\| & \le & \boldsymbol{b}_i, & i = 1, \ldots, m, \\ & A\boldsymbol{x} & \le & \boldsymbol{b}, & \\ & r & \ge & x_j - y_j, & j = 1, \ldots, n, \\ & r & \ge & -(x_j - y_j), & j = 1, \ldots, n. \end{array}$$

3. Let $x_i = 1$ if Investment i is chosen, and 0 otherwise, for $i = 1, \ldots, 7$.
   (a) $\sum_{i=1}^{7} x_i \leq 6$.
   (b) $\sum_{i=1}^{7} x_i \geq 1$.
   (c) $x_2 \leq 1 - x_3$.
   (d) $x_4 \leq x_1$.
   (e) $x_1 + x_2 = 2z$, where $z \in \{0, 1\}$.
   (f) $x_1 + x_2 + x_3 \geq z, \quad x_2 + x_3 + x_5 + x_6 \geq 2(1 - z)$, where $z \in \{0, 1\}$.

4. We want to model the order in which the jobs are scheduled so that we can express the starting time of each job. Hence we will use the following decision variables.

   $x_{ij} = 1$ if job j is done immediately after job i, and 0 otherwise;
   $y_i = 1$ if job i is the first job handled, and 0 otherwise; and
   $z_i =$ starting time of job i.

   Here, x_{ij}, $y_i \in \{0, 1\}$, while $z_i \geq 0$ are continuous variables. We then minimize $z = \sum_{j=1}^{n} w_j z_j$, the weighted sum of starting times. We need to enforce the following.

   - If $x_{ij} = 1$, then $z_j = z_i + p_i$. Writing $z_j \geq z_i + p_i x_{ij} - \left(\sum_{k=1}^{n} p_k\right)(1 - x_{ij})$ will do. Notice that the big-M used here is the sum of the running times of all the jobs.
   - If $y_j = 1$, then $x_{ij} = 0 \ \forall\ i$ (if job j is the first one, then it does not follow any other job). We add $x_{ij} \leq 1 - y_j$.

5. Let $x_{ij} = 1$ if a queen is placed in square (i, j) of the $N \times N$ grid, and 0 otherwise. Apart from the easier constraints (total number of queens should be N, not more than one queen along each row and along each column), we need to enforce the constraints for the two classes of diagonals. For the diagonals going SW–NE, we write the following two constraints.

$$\sum_{k=i}^{N} x_{k,k+1-i} \leq 1 \ \ \forall\ i = 1, \ldots, N; \text{ and}$$

$$\sum_{k=1}^{N+1-j} x_{k,j+k-1} \leq 1 \ \ \forall\ j = 1, \ldots, N.$$

   For the NW–SE diagonals, we write the following two sets of constraints.

$$\sum_{k=1}^{i} x_{k,i+1-k} \leq 1 \ \ \forall\ j = 1, \ldots, N-1; \text{ and}$$

$$\sum_{k=j}^{N} x_{k,N+j-k} \leq 1 \ \ \forall\ j = 1, \ldots, N-1.$$