Introduction to Linear Algebra (Math 220, Section 2) – Fall 2013
Brief Solutions to Practice Midterm Exam

1. **(12)** Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & 0.3x_3 & = & -3 \\ & & x_2 & + & 6x_3 & = & 5 \\ -2x_1 & - & 5x_2 & + & 7x_3 & = & 0 \end{array}$$

   (a) Write the system as a matrix equation.

$$\begin{bmatrix} 3 & 4 & 0.3 \\ 0 & 1 & 6 \\ -2 & -5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}.$$

   (b) Write the system as a vector equation.

$$x_1 \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 0.3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}.$$

   (c) Write the augmented matrix for the system.

$$\left[\begin{array}{rrr|r} 3 & 4 & 0.3 & -3 \\ 0 & 1 & 6 & 5 \\ -2 & -5 & 7 & 0 \end{array}\right]$$

2. **(16)** Let $A = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 3 & 0 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}$.

   (a) Solve the system $A\mathbf{x} = \mathbf{b}$, and write the solution in parametric vector form.

$$\left[\begin{array}{rrrr|r} 1 & 1 & -3 & 1 & 6 \\ 0 & 1 & -2 & 1 & 5 \\ -1 & -1 & 3 & 0 & -3 \end{array}\right] \xrightarrow[R_3 + R_1]{} \left[\begin{array}{rrrr|r} 1 & 1 & -3 & 1 & 6 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_1 - R_2]{} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \implies \begin{array}{l} x_1 = 1 + x_3 \\ x_2 = 2 + 2x_3 \\ x_3 = x_3 \\ x_4 = 3 \end{array}$$

   The parametric vector form of the solutions is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \ s \in \mathbb{R}$.

   (b) Using the result from Part (a), write the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ in the parametric vector form.

   The parametric vector form of the solutions to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \ s \in \mathbb{R}$.

3. **(10)** Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \quad \text{and } \mathbf{w} = \begin{bmatrix} 0.5 \\ 2 \\ -5 \end{bmatrix}.$$

It can shown that $3\mathbf{u} - \mathbf{v} = 2\mathbf{w}$. Use this fact (and *no row operations*) to find a non-trivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 2 & 1 & 0.5 \\ 5 & 3 & 2 \\ 7 & -1 & -5 \end{bmatrix}.$$

Notice that $-\mathbf{v} + 3\mathbf{u} - 2\mathbf{w} = \mathbf{0}$, and hence a non-trivial solution to the homogeneous system is given by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

4. **(12)** Construct a 3×3 matrix A with every entry non-zero such that the following vector $\mathbf{b}$ is ***not*** in the span of the columns of A. Justify your answer.

$$\mathbf{b} = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}$$

We need to create a matrix A such that the system $A\mathbf{x} = \mathbf{b}$ is **inconsistent**. One straightforward way to do this task is to make an A with all three columns being $\mathbf{b}$ itself, and then change all entries in one of the rows to a different number. For example, consider

$$A = \begin{bmatrix} 8 & 8 & 8 \\ -3 & -3 & -3 \\ 2 & 2 & 2 \end{bmatrix}. \text{ Then } [A \,|\, \mathbf{b}] = \begin{bmatrix} 8 & 8 & 8 & | & 8 \\ -3 & -3 & -3 & | & -3 \\ 2 & 2 & 2 & | & 1 \end{bmatrix} \xrightarrow[R_2 + (3/2)R_3]{} \begin{bmatrix} 8 & 8 & 8 & | & 8 \\ 0 & 0 & 0 & | & -3/2 \\ 2 & 2 & 2 & | & 1 \end{bmatrix},$$

showing that the system is inconsistent.

5. **(12)** Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}.$$

(a) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ span $\mathbb{R}^3$? Why or why not?

$$\begin{bmatrix} 3 & 6 & 5 & 5 \\ 1 & 2 & -2 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_1 \leftrightharpoons R_2]{} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 3 & 6 & 5 & 5 \\ 4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 - 3R_1,\ R_3 - 4R_1]{} \begin{bmatrix} \mathbf{1} & 2 & -2 & 0 \\ 0 & \mathbf{-7} & 9 & 2 \\ 0 & 0 & \mathbf{11} & 5 \end{bmatrix}.$$

There is a pivot in every row, and hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ spans $\mathbb{R}^3$

(b) Does $\{\mathbf{v}_1, \mathbf{v}_2\}$ span $\mathbb{R}^3$? Why or why not?

We need a pivot in every row, i.e., we need three pivots. With only two vectors, we can have only two pivots at the best. Hence $\{\mathbf{v}_1, \mathbf{v}_2\}$ does not span $\mathbb{R}^3$.

6. **(11)** Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first does a horizontal shear transformation mapping $\mathbf{e}_2$ to $\mathbf{e}_2 + 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged), and then reflects points through the vertical axis.

$$T(\mathbf{e}_1) = -\mathbf{e}_1, \text{ and } T(\mathbf{e}_2) = \mathbf{e}_2 - 2\mathbf{e}_1. \text{ Hence } T(\mathbf{x}) = A\mathbf{x}, \text{ where } A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}.$$

7. **(15)** Consider the following system.

$$\begin{aligned} x_1 + 3x_2 &= k \\ x_1 - hx_2 &= 2 \end{aligned}$$

Determine all the values of the parameters h and k for which each of the following statements are true.

$$\left[\begin{array}{cc|c} 1 & 3 & k \\ 1 & -h & 2 \end{array}\right] \xrightarrow[R_2 - R_1]{} \left[\begin{array}{cc|c} 1 & 3 & k \\ 0 & -h-3 & 2-k \end{array}\right].$$

(a) The system has no solution.
We need the second row to be $[0\ 0 \mid \blacksquare]$ for the system to be inconsistent. Hence we need $-h-3=0$ and $2-k \neq 0$. Thus, the system has no solution when $h=-3$, and k is any real number other than 2.

(b) The system has a unique solution.
We need two pivots (and hence o free variables). Hence $-h-3 \neq 0$, and k can be any real number.

(c) The system has many solutions.
We need a free variable. Hence, $-h-3=0,\ 2-k=0$, i.e., $h=-3, k=2$.

8. **(12)** Decide whether each of the following statements is *True* or *False*. Justify your answer.

(a) A 3×3 matrix can have more than three echelon forms.
TRUE. We can have four echelon forms.

$$\begin{bmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{bmatrix}, \quad \begin{bmatrix} \blacksquare & 0 & \blacksquare \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{bmatrix}, \quad \begin{bmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare \end{bmatrix}, \text{ and } \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{bmatrix}.$$

(b) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors in $\mathbb{R}^2$ that are not collinear (i.e., they do not lie along the same line), and let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. Then the system $A\mathbf{x} = \mathbf{b}$ cannot have infinitely many solutions for any $\mathbf{b}$.
TRUE. The columns are linearly independent, and hence there is a pivot in every column, so there are no free variables.

(c) If a linear transformation is onto, then it cannot be one-to-one.
FALSE. It can be both 1-to-1 and onto at the same time, e.g., when A is a square matrix with a pivot in each diagonal entry – and hence has a pivot in each row and each column.

(d) If A is an $m \times n$ matrix, the range of the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.
FALSE. The co-domain is $\mathbb{R}^m$. The range can be a subset of $\mathbb{R}^m$.